

INVARIANT DIFFERENTIAL OPERATORS AND AN INFINITE DIMENSIONAL HOWE-TYPE CORRESPONDENCE.

PART I: STRUCTURE OF THE ASSOCIATED ALGEBRAS OF
DIFFERENTIAL OPERATORS

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ABSTRACT. If Q is a non degenerate quadratic form on \mathbb{C}^n , it is well known that the differential operators $X = Q(x)$, $Y = Q(\partial)$, and $H = E + \frac{n}{2}$, where E is the Euler operator, generate a Lie algebra isomorphic to \mathfrak{sl}_2 . Therefore the associative algebra they generate is a quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$. This fact is in some sense the foundation of the metaplectic representation. The present paper is devoted to the study of the case where $Q(x)$ is replaced by $\Delta_0(x)$, where $\Delta_0(x)$ is the relative invariant of a prehomogeneous vector space of commutative parabolic type (\mathfrak{g}, V) , or equivalently where Δ_0 is the "determinant" function of a simple Jordan algebra V over \mathbb{C} . In this Part I we show several structure results for the associative algebra generated by $X = \Delta_0(x)$, $Y = \Delta_0(\partial)$. Our main result shows that if we consider this algebra as an algebra over a certain commutative ring \mathbf{A} of invariant differential operators it is isomorphic to the quotient of what we call a generalized Smith algebra $S(f, \mathbf{A}, n)$ where $f \in \mathbf{A}[t]$. The Smith algebras (over \mathbb{C}) were introduced by P. Smith as "natural" generalizations of $\mathcal{U}(\mathfrak{sl}_2)$. In the forthcoming Part II we will consider the Lie algebra \mathcal{L} generated by X, Y and $\mathfrak{gl}(V)$, the Lie algebra \mathcal{A} generated by X and Y , and put $\mathcal{B} = \mathfrak{g}$ (where \mathfrak{g} is the structure algebra). Then \mathcal{A} and \mathcal{B} are commuting subalgebras of \mathcal{L} . Moreover the restriction of the natural representation of \mathcal{L} on polynomials on V to $\mathcal{A} \times \mathcal{B}$ gives rise to a correspondence between some highest weight modules of \mathcal{A} and the "harmonic" representation of \mathcal{B} , which generalizes the Howe correspondence between highest weight modules of \mathfrak{sl}_2 and ordinary spherical harmonics. The Lie algebras \mathcal{L} and \mathcal{A} are infinite-dimensional except if Δ_0 is a quadratic form, and in this case \mathcal{L} is the usual symplectic algebra, $\mathcal{A} = \mathfrak{sl}_2$ and the above mentioned representation is the infinitesimal metaplectic representation.

1. INTRODUCTION

1.1. Let $Sp(n, \mathbb{R})$ be the real symplectic group on rank n and let $\widetilde{Sp}(n, \mathbb{R})$ be its two fold covering group, the so-called metaplectic group. There is a unitary representation of $Sp(n, \mathbb{R})$, constructed by Shale [Sh] and Weil [We], which we will call the *metaplectic* representation (and denote by π), that is of considerable interest in representation theory. In this paper we will propose an infinitesimal generalization of the metaplectic representation,

which is different from the *minimal representations*. Let us recall some basic facts about this representation and the infinitesimal Howe correspondence between harmonic representations of $\mathfrak{o}(n)$ and some lowest-weight modules of $\mathfrak{sl}(2)$.

Let \tilde{U} be the two fold covering group of the unitary group U in n variables. The group U (respectively \tilde{U}) is the maximal compact subgroup of $Sp(n, \mathbb{R})$ (respectively $\tilde{Sp}(n, \mathbb{R})$). The metaplectic representation π of $\tilde{Sp}(n, \mathbb{R})$ is usually realized in $L^2(\mathbb{R})$. However there exists a realization of the corresponding $(\mathfrak{sp}(n, \mathbb{C}), \tilde{U})$ -module of π , called the Fock model ([H-1]), where the space of \tilde{U} -finite vectors is the space $\mathbb{C}[\mathbb{C}^n]$ of polynomials in n variables.

In order to describe explicitly the corresponding infinitesimal representation π^∞ of the complexified Lie algebra $\mathfrak{sp}(n, \mathbb{C})$, we need first to remark that $\mathfrak{sp}(n, \mathbb{C})$ is 3-graded:

$$\mathfrak{sp}(n, \mathbb{C}) = V^- \oplus \mathfrak{g} \oplus V^+$$

where $V^- \simeq V^+ \simeq \text{Sym}_{\mathbb{C}} =$ the space of symmetric $n \times n$ complex matrices and where $\mathfrak{g} \simeq \mathfrak{gl}(n, \mathbb{C})$. More explicitly

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^t A \end{pmatrix}, A \in \mathfrak{gl}(n, \mathbb{C}) \right\}$$

and

$$V^+ = \left\{ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, X \in \text{Sym}_{\mathbb{C}} \right\} \text{ and } V^- = \left\{ \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}, Y \in \text{Sym}_{\mathbb{C}} \right\}.$$

Let Q_X denote the quadratic form associated to the symmetric matrix X . In other words $Q_X(x) = {}^t x X x$, where x is a column vector in \mathbb{C}^n . Let us also denote by $Q_X(\partial)$ the differential operator with constant coefficients obtained by replacing x_i by $\frac{\partial}{\partial x_i}$. Then in the Fock model we have, for all $P \in \mathbb{C}[\mathbb{C}^n]$ (see [H-1]):

$$\forall X \in V^+, \pi^\infty(X)P(x) = Q_X(x)P(x)$$

$$\forall Y \in V^-, \pi^\infty(Y)P(x) = Q_Y(\partial)P(x)$$

$$\forall A \in \mathfrak{g}, \pi^\infty(A)P(x) = \frac{\text{Tr}(A)}{2}(x) + (A.P)(x) = \frac{\text{Tr}(A)}{2}(x) + P'(x)(Ax) \quad (1.1.1)$$

On the other hand, if the quadratic form Q_X is non degenerate (i.e if $\det(X) \neq 0$), then we consider the embedding of \mathfrak{sl}_2 into $\mathfrak{sp}(n, \mathbb{C})$ defined by

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} a\text{Id} & bX \\ cX^{-1} & -a\text{Id} \end{pmatrix} \quad (1.1.2)$$

From (1.1) and (1.2) we obtain the following \mathfrak{sl}_2 -triple of differential operators:

$$Y = Q_{X^{-1}}(\partial), H = E + \frac{n}{2}, X = Q_X(x) \quad (1.1.3)$$

(Here and in the whole paper we adopt the following convention for \mathfrak{sl}_2 -triples (Y, H, X) :

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [Y, X] = H).$$

Let us denote by $\mathfrak{sl}_{2,X}$ the Lie algebra isomorphic to \mathfrak{sl}_2 generated by Y, H, X . Let us also denote by $\mathfrak{o}(Q_X)$ the orthogonal algebra of Q_X , viewed as a subalgebra of $\mathfrak{g} \hookrightarrow \mathfrak{sp}(n, \mathbb{C})$. It is easy to see that the pair $(\mathfrak{o}(Q_X), \mathfrak{sl}_{2,X})$ is a dual pair, in other words these two subalgebras are mutual centralizers in $\mathfrak{sp}(n, \mathbb{C})$.

One of the most striking fact attached to the metaplectic representation is the Howe correspondence. Suppose that (G_1, G_2) is a reductive dual pair in $Sp(n, \mathbb{R})$ (this means just that the groups G_1 and G_2 act reductively on \mathbb{R}^n and are mutual centralizers in $Sp(n, \mathbb{R})$). Then the pre-images \widetilde{G}_1 and \widetilde{G}_2 under the covering map are again mutual centralizers in $\widetilde{Sp}(n, \mathbb{R})$. For sake of simplicity let us suppose that G_1 is compact and denote by \mathfrak{g}_1 and \mathfrak{g}_2 the corresponding complexified Lie sub-algebras of G_1 and G_2 . Then $\pi^\infty|_{\mathfrak{g}_1 \times \mathfrak{g}_2}$ decomposes multiplicity free (see [H-3], Theorem 4.3 p. 190):

$$\pi^\infty|_{\mathfrak{g}_1 \times \mathfrak{g}_2} = \oplus (V_\sigma \otimes W_{\mu(\sigma)}) \quad (1.1.4)$$

where V_σ is a finite dimensional irreducible module for \mathfrak{g}_1 and $W_{\mu(\sigma)}$ is an irreducible module module for \mathfrak{g}_2 . Moreover the correspondence $\sigma \rightarrow \mu(\sigma)$ is a bijection, the so-called Howe correspondence. For the general result by Howe we refer the reader to [H-1].

In the case of the dual pair $(\mathfrak{o}(Q_X), \mathfrak{sl}_{2,X})$, the Howe correspondence is given by

$$\pi^\infty|_{\mathfrak{o}(Q_X) \times \mathfrak{sl}_{2,X}} = \oplus_{k \in \mathbb{N}} (\mathcal{H}_k \otimes M(k)) \quad (1.1.5)$$

where \mathcal{H}_k is the space of spherical harmonics of degree k and where $M(k)$ is a lowest weight module of $\mathfrak{sl}_{2,X}$ with lowest weight $k + \frac{n}{2}$ (a discrete series representation) See [H-2], Theorem p. 833.

Remark 1.1.1. It is worth noticing that the restriction of π^∞ to $\mathfrak{sl}_{2,X}$ completely determines the metaplectic representation π^∞ . This is due to the fact that the Lie algebra $\mathfrak{sp}(n, \mathbb{C})$ is generated by $\mathfrak{sl}_{2,X}$ and by $\mathfrak{g} \simeq \mathfrak{gl}(n, \mathbb{C})$ and the fact that the action of $\mathfrak{gl}(n, \mathbb{C})$ is the natural action, twisted by the character $\frac{Tr}{2}$ coming from the unitarity of π (see (1.1.1)).

From the preceding remark we see that the infinitesimal metaplectic representation is completely determined by the \mathfrak{sl}_2 -triple (1.1.3). More precisely given a non-degenerate quadratic form Q on \mathbb{C}^n one can reconstruct both the symplectic Lie algebra and the metaplectic representation.

It is natural to ask if there exists such a theory if we replace Q by any homogeneous irreducible polynomial P . The present paper is devoted to the case where Q is replaced by Δ_0 , the fundamental relative invariant of an irreducible regular prehomogeneous vector space of commutative parabolic type (\mathfrak{g}, V) , or equivalently under the Koecher-Tits construction ([Ko],[Ti]), where Δ_0 is the "determinant" function of a simple Jordan algebra over \mathbb{C} . It should also be mentioned that these two categories of objects are equivalent to the hermitian symmetric spaces of tube type.

1.2. This paper splits into two parts. The present first part, covered by sections 2 to 7 is essentially concerned with the study of various algebras of differential operators, in particular the algebra denoted $\mathcal{T}_0[X, Y]$, which is the generalisation of the associative algebra generated by the \mathfrak{sl}_2 -triple

(1.1.3). The forthcoming second part will concern the generalization of the metaplectic representation and an infinite dimensional correspondence which is analogue to the correspondence (1.1.5).

Let us now give an outline of the Part I.

In Section 2 we will briefly recall basic facts concerning Prehomogeneous Vector Spaces (abbreviated PV), more precisely those which are of commutative parabolic type. These objects are in one-to-one correspondence with 3-gradings of simple Lie algebras. In particular we will define the rank of these objects as well as the inductive construction of strongly orthogonal roots which leads to the orbit structure. A key ingredient for the sequel is the multiplicity free decomposition of the polynomials on the PV.

In section 3 we show that the Lie algebra \mathfrak{a} generated by $\Delta_0(x), \Delta_0(\partial)$ and E is infinite dimensional in each of his homogeneous component except if Δ_0 has degree 1 or 2.

In section 4 we show that the associative algebra \mathcal{T} generated by $\Delta_0(x), \Delta_0(x)^{-1}, \Delta_0(\partial)$ and E , which is the biggest algebra of interest to us, can be embedded in the Weyl algebra of a complex one-dimensional torus (that is the algebra of differential operators with regular coefficients) tensored by a polynomial algebra in $r - 1$ variables, where r is the rank.

Various algebras of invariant differentiable operators related to different group actions occuring in our context are defined and studied in section 5. In this section we also introduce the Harish-Chandra isomorphism for the open orbit which is a symmetric space. This Harish-Chandra isomorphism will be an important tool for us.

In section 6 we describe the center $\mathcal{Z}(\mathcal{T})$ of \mathcal{T} and the ideals of \mathcal{T} . We also prove that \mathcal{T} and some other algebras are noetherian and we compute their Gelfand-Kirillov dimension.

Section 7 contains the main result of this first part. We introduce there the so-called generalized Smith algebras $S(\mathbf{R}, f, n)$ where \mathbf{R} is a commutative associative algebra over \mathbb{C} , $f \in \mathbf{R}[t]$ and $n \in \mathbb{N}$. We show that the algebra $\mathcal{T}_0[X, Y]$ (which is the "polynomial" part of \mathcal{T}) is isomorphic to a quotient of such a Smith algebra.

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2. PREHOMOGENEOUS VECTOR SPACES

In this sections we summarize some results and notations we will need about Prehomogeneous Vector Spaces (abbreviated PV).

2.1. Prehomogeneous Vector Spaces. Basic definitions and properties.

For the general theory of PV's, we refer the reader to the book of Kimura [Ki]. Let G be an algebraic group over \mathbb{C} , and let (G, ρ, V) be a rational

representation of G on the (finite dimensional) vector space V . Then the triplet (G, ρ, V) is called a *Prehomogeneous Vector Space* if the action of G on V has a Zariski open orbit $\Omega \in V$. The elements in Ω are called *generic*. The PV is said to be *irreducible* if the corresponding representation is irreducible. The *singular set* S of (G, ρ, V) is defined by $S = V \setminus \Omega$. Elements in S are called *singular*. If no confusion can arise we often simply denote the PV by (G, V) . We will also write $g.x$ instead of $\rho(g)x$, for $g \in G$ and $x \in V$. It is easy to see that the condition for a rational representation (G, ρ, V) to be a PV is in fact an infinitesimal condition. More precisely let \mathfrak{g} be the Lie algebra of G and let $d\rho$ be the derived representation of ρ . Then (G, ρ, V) is a PV if and only if there exists $v \in V$ such that the map:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & V \\ X & \longmapsto & d\rho(X)v \end{array}$$

is surjective (we will often write $X.v$ instead of $d\rho(X)v$). Therefore we will call (\mathfrak{g}, V) a PV if the preceding condition is satisfied.

From now on (G, V) will denote a PV . A rational function f on V is called a *relatively invariant* of (G, V) if there exists a rational character χ of G such that $f(g.x) = \chi(g)P(x)$ for $g \in G$ and $x \in V$. From the existence of an open orbit it is easy to see that a character χ which is trivial on the isotropy subgroup of an element $x \in \Omega$ determines a unique relatively invariant P . Let S_1, S_2, \dots, S_k denote the irreducible components of codimension one of the singular set S . Then there exist irreducible polynomials P_1, P_2, \dots, P_k such that $S_i = \{x \in V \mid P_i(x) = 0\}$. The P_i 's are unique up to nonzero constants. It can be proved that the P_i 's are relatively invariants of (G, V) and any relatively invariant f can be written in a unique way $f = P_1^{n_1} P_2^{n_2} \dots P_k^{n_k}$, where $n_i \in \mathbb{Z}$. The polynomials P_1, P_2, \dots, P_k are called the *fundamental relative invariants* of (G, V) . Moreover if the representation (G, V) is irreducible then there exists at most one irreducible polynomial which is relatively invariant.

The Prehomogeneous Vector Space (G, V) is called *regular* if there exists a relatively invariant polynomial P whose Hessian $H_P(x)$ is nonzero on Ω . If G is reductive, then (G, V) is regular if and only if the singular set S is a hypersurface, or if and only if the isotropy subgroup of a generic point is reductive. If the PV (G, V) is regular, then the contragredient representation (G, V^*) is again a PV .

2.2. Prehomogeneous Vector Spaces of commutative parabolic type.

The PV 's of parabolic type were introduced by the author in [Ru-1] and then developped in his thesis (1982, [Ru-2]). A convenient reference is the book [Ru-3]. The papers [M-R-S] and [R-S-1] contain also parts of the results summarized here. Sato and Kimura ([S-K], [Ki]) gave a complete classification of irreducible regular and so called *reduced* PV 's with a reductive group G (*reduced* stands for a specific representative in a certain equivalence class, the details are not needed here). It turns out that most of these PV 's are of parabolic type. The class of PV 's we are interested in, is a subclass of the full class of parabolic PV 's, the so called PV 's of commutative parabolic type. Let us now give a brief account of the results which we will need later.

Let $\tilde{\mathfrak{g}}$ be a simple Lie algebra over \mathbb{C} satisfying the following two assumptions:

a) There exists a splitting $\tilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+$ which is also a 3-grading:

$$\begin{aligned} [\mathfrak{g}, V^+] &\subset V^+, & [\mathfrak{g}, V^-] &\subset V^-, & [V^-, V^+] &\subset \mathfrak{g} \\ [V^+, V^+] &= \{0\}, & [V^-, V^-] &= \{0\}. \end{aligned}$$

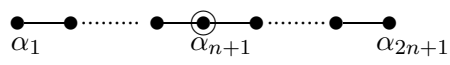
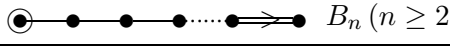
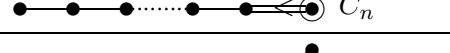
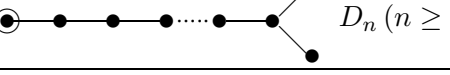

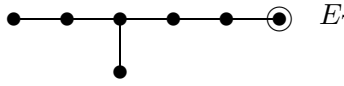
b) There exist a semi-simple element $H_0 \in \mathfrak{g}$ and $X_0 \in V^+, Y_0 \in V^-$ such that (Y_0, H_0, X_0) is an \mathfrak{sl}_2 -triple.

One can prove, that under the assumption a) (\mathfrak{g}, V^+) is an irreducible PV (here the action of \mathfrak{g} on V^+ is the Lie bracket). In fact, as we will sketch now, $\mathfrak{g} \oplus V^+$ is a maximal parabolic subalgebra of $\tilde{\mathfrak{g}}$ whose nilradical V^+ is commutative, this is the reason why these PV's are called of *commutative parabolic type*. Assumption b) is equivalent to the regularity of the PV (\mathfrak{g}, V^+) . We will now describe these PV's in terms of roots. Let \mathfrak{j} be a Cartan subalgebra of \mathfrak{g} which contains H_0 . It is easy to see that \mathfrak{j} is also a Cartan subalgebra of $\tilde{\mathfrak{g}}$. Let \tilde{R} be the set of roots of the pair $(\tilde{\mathfrak{g}}, \mathfrak{j})$. The set P of roots occurring in $\mathfrak{g} \oplus V^+$ is a parabolic subset. Therefore there exists a set of simple roots $\tilde{\Psi}$, such that if \tilde{R}^+ is the corresponding set of positive roots, then $P \subset \tilde{R}^+$. Let ω be the highest root in \tilde{R} and let R be the set of roots of the pair $(\mathfrak{g}, \mathfrak{j})$. Then $\Psi = \tilde{\Psi} \cap R$ is a set of simple roots for R and $\tilde{\Psi} = \Psi \cup \{\alpha_0\}$, where α_0 has coefficient 1 in ω . What we have done up to now can be performed for all 3-gradings of $\tilde{\mathfrak{g}}$ (in other words for all splittings of $\tilde{\mathfrak{g}}$ satisfying the assumption a)). It is easy to see that the element H_0 in assumption b) can be described as the unique element in \mathfrak{j} such that

$$\begin{cases} \alpha(H_0) &= 0 \quad \forall \alpha \in \Psi \\ \alpha_0(H_0) &= 2 \end{cases}$$

Assumption b) means just that H_0 is the semi-simple element of an \mathfrak{sl}_2 -triple. Let w_0 be the unique element of the Weyl group of \tilde{R} such that $w_0(\tilde{\Psi}) = -\Psi$. One can show that the preceding condition on H_0 is equivalent to the condition $w_0(\alpha_0) = -\alpha_0$. This leads to an easy classification of the regular PV's of commutative parabolic type. From the preceding discussion we deduce that these objects are in one to one correspondence with connected Dynkin diagrams where we have circled a root α_0 , which has coefficient 1 in the highest root and such that $w_0(\alpha_0) = -\alpha_0$. In the following table we give the list of these objects and also the corresponding Lie algebra \mathfrak{g} , and the space V^+ .

Table1

	$\tilde{\mathfrak{g}}$	\mathfrak{g}	V^+
A_{2n+1}		$\mathfrak{sl}_{n+1} \times \mathfrak{sl}_{n+1} \times \mathbb{C}$	$M_n(\mathbb{C})$
B_n		$\mathfrak{so}_{2n-2} \times \mathbb{C}$	\mathbb{C}^{2n-2}
C_n		\mathfrak{gl}_n	$MS_n(\mathbb{C})$
D_n^1		$\mathfrak{so}_{2n-1} \times \mathbb{C}$	\mathbb{C}^{2n-1}
D_{2n}^2		$\mathfrak{gl}_{2n}(\mathbb{C})$	$AS_{2n}(\mathbb{C})$
E_7		$E_6 \times \mathbb{C}$	\mathbb{C}^{27}

We need to get more inside the structure of the regular PV 's of commutative parabolic type. First let us define the rank of such a PV . Let \tilde{R}_1 be the set of roots which are orthogonal to α_0 (this is also the set of roots which are strongly orthogonal to α_0). The set \tilde{R}_1 is again a root system as well as $R_1 = \tilde{R}_1 \cap R$. Define

$$\mathfrak{j}_1 = \sum_{\alpha \in \tilde{R}_1} \mathbb{C}H_\alpha, \quad \tilde{\mathfrak{g}}_1 = \mathfrak{j}_1 \oplus \sum_{\alpha \in \tilde{R}_1} \tilde{\mathfrak{g}}^\alpha.$$

Then $\tilde{\mathfrak{g}}_1$ is a semi-simple Lie algebra, \mathfrak{j}_1 is a Cartan subalgebra of $\tilde{\mathfrak{g}}_1$ and the set roots of $(\tilde{\mathfrak{g}}_1, \mathfrak{j}_1)$ is \tilde{R}_1 . Moreover if we set $\mathfrak{g}_1 = \tilde{\mathfrak{g}}_1 \cap \mathfrak{g}$, $V_1^+ = \tilde{\mathfrak{g}}_1 \cap V^+$ and $V_1^- = \tilde{\mathfrak{g}}_1 \cap V^-$, then \mathfrak{j}_1 is also a Cartan subalgebra of \mathfrak{g}_1 and we have

$$\tilde{\mathfrak{g}}_1 = V_1^- \oplus \mathfrak{g}_1 \oplus V_1^+.$$

The key remark is that if we start with $(\tilde{\mathfrak{g}}, \mathfrak{g}, V^+)$ which satisfies assumption a), the preceding splitting of \mathfrak{g}_1 is again a 3-grading satisfying assumptions a), in other words $(\tilde{\mathfrak{g}}_1, \mathfrak{g}_1, V_1^+)$ is again a PV of commutative parabolic type (except that the algebra $\tilde{\mathfrak{g}}_1$ may be semi-simple, not necessarily simple). Moreover if $(\tilde{\mathfrak{g}}, \mathfrak{g}, V^+)$ satisfies also b), then the same is true for $(\tilde{\mathfrak{g}}_1, \mathfrak{g}_1, V_1^+)$.

Let α_1 be the root which plays the role of α_0 for the new PV of commutative parabolic type $(\tilde{\mathfrak{g}}_1, \mathfrak{g}_1, V_1^+)$. We can apply the same procedure, called the *descent*, to $(\tilde{\mathfrak{g}}_1, \mathfrak{g}_1, V_1^+)$ and so on and then we will obtain inductively a sequence

$$\cdots \subset (\tilde{\mathfrak{g}}_k, \mathfrak{g}_k, V_k^+) \subset \cdots \subset (\tilde{\mathfrak{g}}_1, \mathfrak{g}_1, V_1^+) \subset (\tilde{\mathfrak{g}}, \mathfrak{g}, V^+)$$

of PV 's of commutative parabolic type. This sequence stops because, for dimension reasons, there exists an integer n such that $\tilde{R}_n \neq \emptyset$ and such that $\tilde{R}_{n+1} = \emptyset$.

The integer $n + 1$ is then called the *rank* of (\mathfrak{g}, V^+) .

Let us denote by $\alpha_0, \alpha_1, \dots, \alpha_n$ the set of strongly orthogonal roots occurring in V^+ which appear in the descent (the rank is also the number of elements in this sequence, it can be characterized as the maximal number of strongly orthogonal roots occurring in V^+). One proves also that if the first $PV(\mathfrak{g}, V^+)$ is regular (*i.e.* if it satisfies b)), then $V_n^+ = \mathbb{C}X_{\alpha_n}$, where $X_{\alpha_n} \in \tilde{\mathfrak{g}}$, $X_{\alpha_n} \neq 0$.

Let \tilde{G} be the adjoint group of the $\tilde{\mathfrak{g}}$ and let G be the analytic subgroup of \tilde{G} corresponding to \mathfrak{g} . The group G is also the centralizer of H_0 in \tilde{G} . The representation (G, Ad, V^+) is then an irreducible PV . Let G_n be the subgroup of G corresponding to \mathfrak{g}_n . The descent process described before leads to a sequence of PV 's:

$$(G_n, V_n^+) \subset (G_{n-1}, V_{n-1}^+) \subset \dots \subset (G_1, V_1^+) \subset (G, V^+) \quad (2-2-1)$$

The orbital structure of (G, V^+) can be described as follows. Let us denote as usual by X_γ a non zero element of $\tilde{\mathfrak{g}}^\gamma$.

Define

$$\begin{aligned} I_0^+ &= X_{\alpha_0}, I_1^+ = X_{\alpha_0} + X_{\alpha_1}, \dots, I_k^+ = X_{\alpha_0} + X_{\alpha_1} + \dots + X_{\alpha_k}, \dots, \\ I_n^+ &= I^+ = X_{\alpha_0} + X_{\alpha_1} + \dots + X_{\alpha_n}. \end{aligned}$$

Then the set

$$\{0, I_0^+, I_1^+, \dots, I_n^+ = I^+\}$$

is a set of representatives of the G -orbits in V^+ (there are $\text{rank}(G, V^+) + 1$ orbits). The orbit $G.I^+$ is the open orbit $\Omega^+ \subset V^+$.

The Killing form \tilde{B} of $\tilde{\mathfrak{g}}$ allows us to identify V^- to the dual space of V^+ and the representation (G, V^-) becomes then the dual PV of (G, V^+) . One can similarly perform a descent on the V^- side, and obtain a sequence

$$(G_n, V_n^-) \subset (G_{n-1}, V_{n-1}^-) \subset \dots \subset (G_1, V_1^-) \subset (G, V^-) \quad (2-2-2)$$

of PV 's, where the groups are the same as in $(2-2-1)$. The $PV(G_i, V_i^-)$ is dual to (G_i, V_i^+) . The set of elements

$$\{0, I_0^-, I_1^-, \dots, I_n^- = I^-\}$$

where $I_i^- = X_{-\alpha_0} + X_{-\alpha_1} + \dots + X_{-\alpha_i}$ is a set of representatives of the G -orbits in V^- . The orbit $G.I^-$ is the open orbit $\Omega^- \subset V^-$. We will always choose the elements $X_{-\alpha_i}$ such that $(X_{-\alpha_i}, H_{\alpha_i}, X_{\alpha_i})$ is a \mathfrak{sl}_2 -triple. If the $PV(\mathfrak{g}, V^+)$ satisfies the assumptions a) and b), then $H_0 = H_{\alpha_0} + H_{\alpha_1} + \dots + H_{\alpha_n}$ and (I^-, H_0, I^+) is a \mathfrak{sl}_2 -triple. More generally, under the same hypothesis, if $H_i = H_{\alpha_0} + H_{\alpha_1} + \dots + H_{\alpha_i}$, then (I_i^-, H_i, I_i^+) is a \mathfrak{sl}_2 -triple.

We suppose from now on that (G, V^+) is regular. Remember that this means that it satisfies assumption b). Let Δ_0 be the unique irreducible polynomial on V^+ which is relatively invariant under the action of G . Let Δ_1 be the unique irreducible polynomial on V_1^+ which is relatively invariant under the group G_1 . We have $V^+ = W_1^+ \oplus V_1^+$ where W_1^+ is the sum of the root spaces of V^+ which do not occur in V_1^+ . Therefore for $x = y_1 + x_1$ ($x_1 \in V_1^+, y_1 \in W_1^+$), we can define $\Delta_1(x) = \Delta_1(x_1)$ and hence the polynomial $\Delta_1(x)$ can be viewed as a polynomial on V^+ . Inductively we can define a sequence $\Delta_0, \Delta_1, \dots, \Delta_n$ of irreducible polynomials on V^+ ,

where the polynomial Δ_i depends only on the variables in V_i^+ and is, in general, not relatively invariant under G , but under $G_i \subset G$. It can be shown that $\partial^\circ(\Delta_i) = n + 1 - i = \text{rank}(G, V^+) - i$.

Let $H \subset G$ be the isotropy subgroup of I^+ and let $\mathfrak{h} \subset \mathfrak{g}$ be its Lie algebra. Another striking fact concerning the irreducible regular PV 's of commutative parabolic type is that the open orbit $G/H \simeq \Omega^+$ is a symmetric space. This means that \mathfrak{h} is the fixed points set of an involution of \mathfrak{g} (this involution can be shown to be $\exp \text{ad}(I^+) \exp \text{ad}(I^-) \exp \text{ad}(I^+)$). Let us denote by B^- the Borel subgroup of G defined by $-\Psi$. One can show that V^+ is already a PV under the action of B^- . More precisely the fundamental relative invariants of (B^-, V^+) are the polynomials $\Delta_0, \Delta_1, \dots, \Delta_n$. The open B^- -orbit in V^+ is the set

$$\mathcal{O}^+ = \{x \in V^+ \mid \Delta_0(x)\Delta_1(x)\dots\Delta_n(x) \neq 0\}.$$

Symmetrically, if B^+ is the Borel subgroup of G defined by Ψ , the representation (B^+, V^-) is also a PV , and the fundamental relatively invariant polynomials is a set

$$\{\Delta_0^*, \Delta_1^*, \dots, \Delta_n^*\}$$

of irreducible polynomials where $\partial^\circ(\Delta_i^*) = n + 1 - i = \text{rank}(G, V^+) - i$. Of course these polynomial are obtained by a descent process similar to the one described before on V^+ . Moreover the polynomial Δ_0^* is the fundamental relatively invariant of (G, V^-) .

Let \tilde{B} be the Killing form on $\tilde{\mathfrak{g}}$. Then for any polynomial P^* on V^- , we define a differential operator $P^*(\partial)$ with constant coefficients on V^+ by the formula

$$P^*(\partial)e^{\tilde{B}(x,y)} = P(y)e^{\tilde{B}(x,y)} \text{ for } x \in V^+, y \in V^-. \quad (2-2-3)$$

The differential operators $\Delta_0^*(\partial)$ and $\Delta_0(x)$ (multiplication by the function $\Delta_0(x)$) will play an important role in this paper. From the fact that (B^-, V^+) has an open orbit it is easy to see that the space $\mathbb{C}[V^+]$ of polynomialson V^+ , viewed as a representation space of G decomposes multiplicity free. For $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{N}^{n+1}$, let $V_{\mathbf{a}}$ be the G -submodule of $\mathbb{C}[V^+]$ generated by $\Delta_{\mathbf{a}}(x) = \Delta_0^{a_0}(x)\Delta_1^{a_1}(x)\dots\Delta_n^{a_n}(x)$. Then $V_{\mathbf{a}}$ is an irreducible G -module with highest weight vector $\Delta_{\mathbf{a}}(x)$ (with respect to $-\Psi$). Moreover the representations $V_{\mathbf{a}}$ and $V_{\mathbf{a}'}$ are not equivalent if $\mathbf{a} \neq \mathbf{a}'$ and

$$\mathbb{C}[V^+] = \oplus_{\mathbf{a} \in \mathbb{N}^{n+1}} V_{\mathbf{a}} \quad (2-2-4)$$

Remark 2.2.1. Let G/K be a hermitian symmetric space, let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^- \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}^+$ be the usual decomposition of the complexified Lie algebra of G . Then of course the preceding slitting is a 3-grading of $\mathfrak{g}_{\mathbb{C}}$. Moreover this 3-grading verifies also assumption b) at the beginning of section 2.2. if and only if G/K is of tube type. Conversely it can be shown that any PV of commutative parabolic type can be obtained this way from a hermitian symmetric space of tube type.

Similarly if J is a simple Jordan algebra over \mathbb{C} , and if $\text{struc}(J)$ is its structure algebra, then from the Koecher-Tits construction ([Ko], [Ti]) it is know that one can put a Lie algebra structure on $J \oplus \text{struc}(J) \oplus J$, and this splitting is a 3-grading which verifies assumption b). Conversely if (G, V^+)

is a PV of commutative parabolic type, then one can define on V^+ a Jordan product which makes V^+ into simple Jordan algebra (see also [F-K]).

3. SOME ALGEBRAS OF DIFFERENTIAL OPERATORS

3.1. Definitions and basic properties.

For any smooth affine algebraic variety M , we shall denote by $\mathbb{C}[M]$ the algebra of regular functions on M . If $M = U$ is a vector space, then $\mathbb{C}[U]$ is the algebra of polynomials on U . As $S^+ = V^+ \setminus \Omega^+$ is the hypersurface defined by Δ_0 , the algebra $\mathbb{C}[\Omega^+]$ of regular functions on Ω^+ is the algebra of fractions of the form $\frac{P}{\Delta_0^k}$, $P \in \mathbb{C}[V^+]$.

We shall denote by $\mathbf{D}(M)$ the algebra of differential operators on M . For example if U is a vector space, the algebra $\mathbf{D}(U)$ is the Weyl algebra of U . If D_1, D_2, \dots, D_k is a finite family of elements in $\mathbf{D}(M)$, we will denote by $\mathbb{C}[D_1, D_2, \dots, D_k]$ the subalgebra (with unit) of $\mathbf{D}(M)$ generated by D_1, D_2, \dots, D_k . Of course $\mathbb{C}[D_1, D_2, \dots, D_k]$ is also the set of linear combinations of monomials in the noncommutative generators D_1, D_2, \dots, D_k .

If g is any algebraic diffeomorphism of the variety M , and $f \in \mathbb{C}[M]$, we define the function $\tau_g f \in \mathbb{C}[M]$ by $\tau_g f(x) = f(g^{-1}x)$, $x \in M$. Then for any $D \in \mathbf{D}(M)$, we also define $\tau_g D \in \mathbf{D}(M)$ by $\tau_g D = \tau_g \circ D \circ \tau_g^{-1}$.

We define two differential operators in $\mathbf{D}(V^+)$ by

$$\begin{aligned} X &= \Delta_0(x) && \text{(multiplication by } \Delta_0(x)) \\ Y &= \Delta_0^*(\partial). && \text{(defined by (2-2-3))} \end{aligned}$$

We will also consider the Euler operator E on V^+ which is defined on any $P \in \mathbb{C}[V^+]$, by

$$EP(x) = \frac{\partial}{\partial t} P(tx)_{t=1} = P'(x)x.$$

Let us recall from [R-S-1] page 424, rel(4-4), that if χ_0 is the character of Δ_0 , we have:

$$\begin{aligned} \forall g \in G, \quad \tau_g X &= \chi_0(g^{-1})X \\ \tau_g Y &= \chi_0(g)Y && (3-1-1) \\ \tau_g E &= E. \end{aligned}$$

Let \mathfrak{a} be the Lie sub-algebra of $\mathbf{D}(V^+)$ generated by X , Y and E and let \mathcal{A} be the associative sub-algebra of $\mathbf{D}(V^+)$ generated by X , Y , E and \mathbb{C} ($\mathcal{A} = \mathbb{C}[X, Y, E]$).

The algebra $\mathbf{D}(\Omega^+)$ is the algebra of differential operators on V^+ whose coefficients lie in $\mathbb{C}[\Omega^+]$. Of course one has $\mathbf{D}(V^+) \subset \mathbf{D}(\Omega^+)$.

We also introduce another differential operator $X^{-1} \in \mathbf{D}(\Omega^+)$:

$$X^{-1} = \Delta_0(x)^{-1} \quad (\text{multiplication by } \Delta_0(x)^{-1}).$$

Let \mathcal{T} be the associative sub-algebra of $\mathbf{D}(\Omega^+)$ generated by X, Y, E, X^{-1} and \mathbb{C} ($\mathcal{T} = \mathbb{C}[E, X, X^{-1}, Y]$). The inclusions

$$\mathfrak{a} \subset \mathcal{A} \subset \mathcal{T}$$

are obvious.

Remember that the space $\mathbb{C}[V^+]$ decomposes multiplicity free (see (2-2-4)) under the action of G :

$$\mathbb{C}[V^+] = \bigoplus_{\mathbf{a} \in \mathbb{N}^{n+1}} V_{\mathbf{a}} \quad (3-1-2)$$

where $V_{\mathbf{a}} = V_{a_0, \dots, a_n}$ is the irreducible G -submodule generated by $\Delta_0^{a_0} \dots \Delta_n^{a_n} = \Delta^{\mathbf{a}}$. Sometimes it will be convenient to use the following notations. If $\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{N}^{n+1}$, then we will write $\mathbf{a} = (a_0, \tilde{\mathbf{a}})$, where $\tilde{\mathbf{a}} = (a_1, \dots, a_n) \in \mathbb{N}^n$, and we will denote $V_{(0, \tilde{\mathbf{a}})}$ by $V_{\tilde{\mathbf{a}}}$.

It is easily seen that $V_{\mathbf{a}} = \Delta_0^{a_0} V_{0, a_1, \dots, a_n} = \Delta_0^{a_0} V_{\tilde{\mathbf{a}}}$.

More generally the space $\mathbb{C}[\Omega^+]$ also decomposes multiplicity free under G :

$$\mathbb{C}[\Omega^+] = \bigoplus_{\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^n} V_{\mathbf{a}} \quad (3-1-3)$$

with the same definition for the spaces $V_{\mathbf{a}}$.

We make the convention that if $\mathbf{z} = (z_0, \dots, z_n)$ is any set of variables, then for any integer $p \in \mathbb{Z}$, $\mathbf{z} + p = (z_0 + p, z_1, \dots, z_n)$ and $\tilde{\mathbf{z}} = (z_1, \dots, z_n)$.

It is easy to see that the operator X maps $V_{\mathbf{a}}$ onto $V_{\mathbf{a}+1}$ and is a G' and \mathfrak{g}' equivariant isomorphism, where G' is the derived group of G and \mathfrak{g}' its Lie algebra. The same way the operator Y maps $V_{\mathbf{a}}$ into $V_{\mathbf{a}-1}$ and is G' and \mathfrak{g}' equivariant ([R-S-1], Lemme 4.3 page 424).

If $P \in V_{\mathbf{a}}$, then $YP = b_Y(\mathbf{a}) \frac{P}{\Delta_0} = b_Y(\mathbf{a}) X^{-1} P$ ([R-S-1], Remarque 4.5.), where b_Y is a polynomial in $n+1$ variables, which we will call the *Bernstein-Sato polynomial* or the *b-function* of the operator Y . If $\mathbf{X} = (X_0, \dots, X_n)$ then we have, under a suitable normalization ¹

$$b_Y(\mathbf{X}) = \prod_{j=0}^n (X_0 + \dots + X_j + j \frac{d}{2}), \quad (3-1-4)$$

where $\frac{d}{2} = \frac{k-(n+1)}{n(n+1)}$, $k = \dim V^+$ (it can be noted that this constant is the same as the constant also denoted by d in the theory of simple Jordan algebras over \mathbb{C} , cf. Remark 2.2.1).

This explicit computation of the polynomial b_Y has been obtained by several authors, using distinct methods (see [B-R], [F-K], [K-S], [Wa]). As a consequence of this, it is easy to see that the operator Y has a kernel $\mathcal{H}[V^+]$

¹The polynomials Δ_0 and Δ_0^* are only defined up to nonzero constants. We suppose here that $\Delta_0(I^+) = 1$ and then we choose Δ_0^* such that (3-1-4) holds

(the "harmonic" polynomials) in $\mathbb{C}[V^+]$ which can be described the following way:

$$\mathcal{H}[V^+] = \bigoplus_{\mathbf{a} \in \{0\} \times \mathbb{N}^n} V_{\mathbf{a}} = \bigoplus_{\tilde{\mathbf{a}} \in \mathbb{N}^n} V_{\tilde{\mathbf{a}}}. \quad (3-1-5)$$

The preceeding decomposition has been obtained, without the explicit knowledge of b_Y , by Rubenthaler-Schiffmann ([R-S-1] Théorème 4.4.) and Upmeyer ([Up], Theorem 2.6.). This decomposition means that the restriction of Y to $V_{\mathbf{a}}$ is an isomorphism onto $V_{\mathbf{a}-1}$ except if $\mathbf{a} \in \{0\} \times \mathbb{N}^n$.

We will now define \mathbb{Z} -gradings on \mathfrak{a} , \mathcal{A} and \mathcal{T} . Recall that the operators E , X , X^{-1} and Y act naturally on the space $\mathbb{C}[\Omega^+]$. Let $\mathbb{C}[V^+] = \bigoplus_{n=0}^{\infty} \mathbb{C}^n[V^+]$ be the natural grading of the polynomials by the homogeneous degree. The homogeneous degree also defines a grading on the regular functions on Ω^+ : $\mathbb{C}[\Omega^+] = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}^n[\Omega^+]$.

Define for each $p \in \mathbb{Z}$:

$$\begin{aligned} \mathfrak{a}_p &= \{D \in \mathfrak{a} \mid D : V_{\mathbf{a}} \mapsto V_{\mathbf{a}+p}, \forall \mathbf{a} \in \mathbb{N}^{n+1}\} \\ &= \{D \in \mathfrak{a} \mid D : \mathbb{C}^m[\Omega^+] \mapsto \mathbb{C}^{m+(n+1)p}[\Omega^+], \forall m \in \mathbb{N}\} \\ &= \{D \in \mathfrak{a} \mid [E, D] = p(n+1)D\} \\ \mathcal{A}_p &= \{D \in \mathcal{A} \mid D : V_{\mathbf{a}} \mapsto V_{\mathbf{a}+p}, \forall \mathbf{a} \in \mathbb{N}^{n+1}\} \\ &= \{D \in \mathcal{A} \mid D : \mathbb{C}^m[\Omega^+] \mapsto \mathbb{C}^{m+(n+1)p}[\Omega^+], \forall m \in \mathbb{N}\} \\ &= \{D \in \mathcal{A} \mid [E, D] = p(n+1)D\} \\ \mathcal{T}_p &= \{D \in \mathcal{T} \mid D : V_{\mathbf{a}} \mapsto V_{\mathbf{a}+p}, \forall \mathbf{a} \in \mathbb{N}^{n+1}\} \\ &= \{D \in \mathcal{T} \mid D : \mathbb{C}^m[\Omega^+] \mapsto \mathbb{C}^{m+(n+1)p}[\Omega^+], \forall m \in \mathbb{N}\} \\ &= \{D \in \mathcal{T} \mid [E, D] = p(n+1)D\} \end{aligned} \quad (3-1-6)$$

Of course one has $E \in \mathfrak{a}_0$, $X \in \mathfrak{a}_1$, $X^{-1} \in \mathcal{T}_{-1}$ and $Y \in \mathfrak{a}_{-1}$. Moreover

$$\mathfrak{a} = \bigoplus_{p \in \mathbb{N}} \mathfrak{a}_p, \quad \mathcal{A} = \bigoplus_{p \in \mathbb{N}} \mathcal{A}_p, \quad \mathcal{T} = \bigoplus_{p \in \mathbb{N}} \mathcal{T}_p \quad (3-1-7)$$

and these decompositions are \mathbb{Z} -gradings of Lie algebras or associative algebras.

An element D belonging to \mathcal{T}_p defines a \mathfrak{g}' -equivariant map from $V_{\mathbf{a}}$ to $V_{\mathbf{a}+p}$. As $V_{\mathbf{a}}$ and $V_{\mathbf{a}+p}$ are irreducible \mathfrak{g}' -modules, there exists a constant $b_D(\mathbf{a})$ such that for each $P \in V_{\mathbf{a}}$, we have $DP = b_D(\mathbf{a})\Delta_0^p P$. It is easy to see that b_D is a polynomial (in $n+1$ variables).

Definition 3.1.1. *If D is a differential operator in \mathcal{T}_p the polynomial $b_D(s)$ defined before is called the Bernstein-Sato polynomial or the b -function of the operator D .*

3.2. First results concerning \mathfrak{a} , \mathcal{A} , and \mathcal{T} .

Theorem 3.2.1. *If the degree of the polynomial Δ_0 is not equal to 1 or 2 (that is if $n > 1$), then the Lie algebra \mathfrak{a} is infinite dimensional. More*

precisely for any $p \in \mathbb{N}$ the space \mathfrak{a}_p is infinite dimensional. As a consequence \mathcal{A}_p and \mathcal{T}_p are also infinite dimensional. If Δ_0 is a quadratic form, then the Lie algebra \mathfrak{a} is isomorphic to a semi-direct product $\mathfrak{sl}_2(\mathbb{C}) \rtimes \mathfrak{c}$, where \mathfrak{c} is one dimensional central. If Δ_0 is of degree one then the Lie algebra \mathfrak{a} is isomorphic to the semi-direct product $H_3 \rtimes \mathbb{C}E$ where H_3 is the three dimensional Heisenberg Lie algebra generated by X and Y .

Proof.

If the degree of $\Delta_0 = 1$, then $\dim V^+ = 1$ and it is clear that the two operators x and $\frac{\partial}{\partial x}$ generate the three dimensional Heisenberg Lie algebra. If Δ_0 is a quadratic form it is also well known that $[Y, X] = E + \frac{k}{2}$ ($k = \dim V^+$) and therefore the Lie algebra generated by X and Y is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. (see for example [Ra-S] or [H-3], p. 199). This fact is in some sense the key point of the construction of the Weil representation. It implies the result for this case.

Suppose now that the degree of Δ_0 is ≥ 3 .

For any $D \in \mathcal{A}$ we shall denote by $\partial^\circ b_D$ the degree of b_D in the a_0 variable. The formula (3-1-4) shows that $\partial^\circ b_Y = n+1$. Let $D \in \mathcal{A}_0$ and suppose that $\partial^\circ b_D = p$, define $\tilde{D} = [Y, D] \in \mathcal{A}_{-1}$. Let $P \in V_{\mathbf{a}}$. We have

$$\begin{aligned} \tilde{D}P &= YDP - DYP = (b_Y(\mathbf{a})b_D(\mathbf{a}) - b_D(\mathbf{a}-1)b_Y(\mathbf{a}))\frac{P}{\Delta_0} \\ &= b_Y(\mathbf{a})(b_D(\mathbf{a}) - b_D(\mathbf{a}-1))\frac{P}{\Delta_0} \end{aligned} \quad (3.2.1)$$

This proves that $b_{\tilde{D}} = b_Y(\mathbf{a})(b_D(\mathbf{a}) - b_D(\mathbf{a}-1))$ and therefore $\partial^\circ b_{\tilde{D}} = p+n$.

Let $\tilde{\tilde{D}} = [X, [Y, D]] = [X, \tilde{D}]$. Let us compute $\tilde{\tilde{D}}P$ for $P \in V_{\mathbf{a}}$. We get:

$$\tilde{\tilde{D}}P = X\tilde{D}P - \tilde{D}XP = (b_{\tilde{D}}(\mathbf{a}) - b_{\tilde{D}}(\mathbf{a}+1))P.$$

Therefore $b_{\tilde{\tilde{D}}}(\mathbf{a}) = b_{\tilde{D}}(\mathbf{a}) - b_{\tilde{D}}(\mathbf{a}+1)$ and hence $\partial^\circ b_{\tilde{\tilde{D}}} = p+n-1$.

Remark that if $n = 1$ (i.e. if Δ_0 is a quadratic form), then $\partial^\circ b_{\tilde{\tilde{D}}} = \partial^\circ b_D$ and if $n = 0$ (i.e. if Δ_0 is a linear form), then $\partial^\circ b_{\tilde{\tilde{D}}} = \partial^\circ b_D - 1$.

Define now inductively the operators $H_q \in \mathfrak{a}_0$ by $H_1 = [X, Y]$, $H_q = [X, [Y, H_{q-1}]]$. As $b_{H_1}(\mathbf{a}) = b_Y(\mathbf{a}) - b_Y(\mathbf{a}+1)$ has degree n in a_0 , one has $\partial^\circ b_{H_q} = (q-1)(n-1) + n$.

Therefore there exists operators in \mathfrak{a}_0 whose Bernstein-Sato polynomial has an arbitrarily large degree in a_0 . Hence $\dim \mathfrak{a}_0 = +\infty$.

Let $p > 0$ and let $D \in \mathfrak{a}_p$. Define $D_1 = [X, D]$. An easy computation shows that $b_{D_1}(\mathbf{a}) = b_D(\mathbf{a}) - b_D(\mathbf{a}+1)$. As we have just proved that there exists operators $D \in \mathfrak{a}_0$ such $\partial^\circ b_D$ is arbitrarily large, the preceding calculation shows inductively that in \mathfrak{a}_p ($p > 0$) too, there exists operators whose Bernstein-Sato polynomial has an arbitrarily large degree in a_0 . Therefore $\dim \mathfrak{a}_p = +\infty$ for $p > 0$.

If $p \leq 0$ and if $D \in \mathfrak{a}_p$, define $D_1 = [Y, D] \in \mathfrak{a}_{p-1}$. An easy computation shows that $b_{D_1}(\mathbf{a}) = b_Y(\mathbf{a})(b_D(\mathbf{a}) - b_D(\mathbf{a}-1))$. The same argument as before proves that $\dim \mathfrak{a}_p = +\infty$.

□

Remark 3.2.2.² The fact that $\dim \mathfrak{a} = +\infty$ (but not the assertion concerning $\dim \mathfrak{a}_p$) is also a consequence of a result by Igusa ([Ig]). Our proof is different from his.

Proposition 3.2.3. *The Lie algebra \mathfrak{a}_0 , as well as the associative algebras \mathcal{A}_0 and \mathcal{T}_0 are commutative.*

Proof. It is enough to prove the result for \mathcal{T}_0 . Let D_1 and D_2 be two operators in \mathcal{T}_0 with Bernstein-Sato polynomials b_{D_1} and b_{D_2} respectively. An easy calculation shows that for any \mathbf{a} , $b_{[D_1, D_2]}(\mathbf{a}) = b_{D_1}(\mathbf{a})b_{D_2}(\mathbf{a}) - b_{D_2}(\mathbf{a})b_{D_1}(\mathbf{a}) = 0$. Therefore $[D_1, D_2] = 0$. \square

4. EMBEDDINGS OF \mathfrak{a} , \mathcal{A} AND \mathcal{T} INTO $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_n]$.

4.1. Vector space embedding of $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n]$ into $\text{End}(\mathbb{C}[\Omega^+])$.

Recall from (3-1-3) the following multiplicity free decomposition of $\mathbb{C}[\Omega^+]$ into irreducible representations of G :

$$\mathbb{C}[\Omega^+] = \bigoplus_{\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^n} V_{\mathbf{a}}.$$

If $P \in V_{\mathbf{a}}$, then

$$XP = \Delta_0 P \in V_{\mathbf{a}+1} \quad (4-1-1)$$

$$YP = b_Y(\mathbf{a}) \frac{P}{\Delta_0} = b_Y(\mathbf{a}) X^{-1} P \in V_{\mathbf{a}-1} \quad (4-1-2)$$

$$EP = b_E(\mathbf{a}) P \in V_{\mathbf{a}} \quad (4-1-3)$$

where we know from (3-1-4) that

$$b_Y(\mathbf{X}) = \prod_{j=0}^n (X_0 + \dots + X_j + j \frac{d}{2})$$

and where

$$b_E(\mathbf{X}) = (n+1)X_0 + nX_1 + \dots + X_n \quad (4-1-4).$$

Therefore the operators E, X, Y, X^{-1} are very well understood as elements of $\text{End}(\mathbb{C}[\Omega^+])$. More precisely the operator E is diagonalisable on $\mathbb{C}[\Omega^+]$, the eigenspaces are the spaces $V_{\mathbf{a}}$ and E acts on each $V_{\mathbf{a}}$ by multiplication by the value at \mathbf{a} of the polynomial b_E . Similarly the action of Y "goes down" from $V_{\mathbf{a}}$ to $V_{\mathbf{a}-1}$. More precisely the action of Y on $V_{\mathbf{a}}$ is the action of X^{-1} (division by Δ_0) together with the multiplication by the value of the polynomial b_Y at \mathbf{a} .

The preceeding remarks suggest that there is an embedding of the vector space $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n]$ into $\text{End}(\mathbb{C}[\Omega^+])$. Let us make this more precise.

²We thank Thierry Levasseur for pointing out the result of Igusa.

If U and V are vector spaces over \mathbb{C} we will denote by $\mathcal{L}(U, V)$ the vector space of linear maps from U to V .

Let us also remark that from the preceeding decomposition of $\mathbb{C}[\Omega^+]$ we get:

$$\text{End}(\mathbb{C}[\Omega^+]) = \bigoplus_{\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^n} \mathcal{L}(V_{\mathbf{a}}, \mathbb{C}[\Omega^+]) \quad (4-1-5)$$

Definition 4.1.1. For $\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^n$, $P \in \mathbb{C}[X_1, \dots, X_n]$, $q \in \mathbb{C}[t, t^{-1}]$ we will denote by $\varphi_{\mathbf{a}}(q \otimes P)$ the element of $\mathcal{L}(V_{\mathbf{a}}, \mathbb{C}[\Omega^+])$ defined by:

$$Q_{\mathbf{a}} \in V_{\mathbf{a}}, \quad \varphi_{\mathbf{a}}(q \otimes P)Q_{\mathbf{a}} = P(\mathbf{a})q(\Delta_0)Q_{\mathbf{a}}$$

($\varphi_{\mathbf{a}}$ defines a linear map from

$$\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n] \text{ into } \mathcal{L}(V_{\mathbf{a}}, \text{End}(\mathbb{C}[\Omega^+]))$$

by the universal property of tensor products).

We then define, using (4-1-5), an element

$$\varphi \in \mathcal{L}(\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n], \text{End}(\mathbb{C}[\Omega^+])) \text{ by}$$

$$\varphi = \bigoplus_{\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^n} \varphi_{\mathbf{a}}.$$

Proposition 4.1.2. The linear map :

$$\varphi : \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n] \longrightarrow \text{End}(\mathbb{C}[\Omega^+])$$

is injective and its image is stable under the multiplication (composition of mappings) in $\text{End}(\mathbb{C}[\Omega^+])$.

Proof. Any element u in $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n]$ can be written as a finite sum

$$u = \sum t^i \otimes P_i \quad i \in \mathbb{Z}, P_i \in \mathbb{C}[X_0, \dots, X_n].$$

Suppose that $u \in \ker \varphi$. Then for any $Q_{\mathbf{a}} \in V_{\mathbf{a}}$ we have:

$$0 = \varphi(u)Q_{\mathbf{a}} = \sum \varphi(t^i \otimes P_i)Q_{\mathbf{a}} = \sum P_i(\mathbf{a})\Delta_0^i Q_{\mathbf{a}}.$$

This implies that $P_i(\mathbf{a}) = 0$ for all i and all \mathbf{a} , and therefore φ is injective.

To prove the stability under multiplication it suffices to prove that for any $R, S \in \mathbb{C}[X_0, \dots, X_n]$ and $\ell, m \in \mathbb{Z}$ the endomorphism $\varphi(t^m \otimes R)\varphi(t^\ell \otimes S)$ belongs to the image of φ . If $Q_{\mathbf{a}} \in V_{\mathbf{a}}$, we have $\varphi(t^m \otimes R)\varphi(t^\ell \otimes S)Q_{\mathbf{a}} = \varphi(t^m \otimes R)S(\mathbf{a})\Delta_0^\ell Q_{\mathbf{a}} = R(\mathbf{a} + \ell)S(\mathbf{a})\Delta_0^{m+\ell} Q_{\mathbf{a}} = \varphi(t^{m+\ell} \otimes (\tau_{-\ell} R)S)Q_{\mathbf{a}}$ where $\tau_\ell R(X) = R(X - \ell)$ and where $X - \ell = (X_0 - \ell, X_1, \dots, X_n)$. This proves that the image is stable under multiplication. \square

4.2. Algebra embedding of $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_n]$ into $\text{End}(\mathbb{C}[\Omega^+])$.

Recall that if \mathbf{A} and \mathbf{B} are two associative algebras the tensor product $\mathbf{A} \otimes \mathbf{B}$ is again an algebra, called the tensor product algebra, with the multiplication defined by

$$a_1, a_2 \in \mathbf{A}, b_1, b_2 \in \mathbf{B} \quad (a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2 \quad (4-2-1)$$

The tensor product algebra $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n]$ is commutative. On the other hand the algebra $\text{End}(\mathbb{C}[\Omega^+])$ is of course non commutative.

The linear injection φ defined in the preceeding section is therefore not an algebra homomorphism. But we will use φ to define a new multiplication in $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n]$ by setting for $q, r \in \mathbb{C}[t, t^{-1}]$ and for $P, Q \in \mathbb{C}[X_0, \dots, X_n]$:

$$(q \otimes P)(r \otimes Q) = \varphi^{-1}(\varphi(q \otimes P)\varphi(r \otimes Q)) \quad (4-2-2)$$

More explicitly it is easy to see that for $m, l \in \mathbb{Z}$ and $P, Q \in \mathbb{C}[X_0, \dots, X_n]$ we have:

$$(t^m \otimes P)(t^l \otimes Q) = t^{m+l} \otimes (\tau_{-l}P)Q \quad (4-2-3)$$

With this multiplication $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n]$ becomes a non commutative associative algebra.

We will denote by $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] = \mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ the algebra of differential operators on \mathbb{C}^* whose coefficients are Laurent polynomials. In fact, using the notation introduced at the beginning of the section 3, $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}]$ is $\mathbf{D}(\mathbb{C}^*)$, the Weyl algebra of the torus.

Proposition 4.2.1. *The algebra $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n]$ whose multiplication is defined by (4-2-3) is isomorphic to the tensor product algebra $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_n]$ (extended Weyl algebra of the torus).*

Proof. We define first an isomorphism

$$\nu : \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n] \longrightarrow \mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_n]$$

between the underlying vector spaces. After that we will prove that ν is in fact an algebra isomorphism.

A vector basis of $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n]$ is given by the elements $t^m \otimes X_0^{\alpha_0} X_1^{\alpha_1} \dots X_n^{\alpha_n}$ ($m \in \mathbb{Z}, \alpha_i \in \mathbb{N}$). Let us define a linear map:

$$\nu(t^m \otimes X_0^{\alpha_0} X_1^{\alpha_1} \dots X_n^{\alpha_n}) = t^m (t \frac{d}{dt})^{\alpha_0} \otimes X_1^{\alpha_1} \dots X_n^{\alpha_n} \quad (4-2-4)$$

On the other hand the elements $t^m (t \frac{d}{dt})^{\alpha_0} \otimes X_1^{\alpha_1} \dots X_n^{\alpha_n}$ define a vector basis of $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_n]$. Therefore the linear map ν is effectively a vector space isomorphism.

Let us recall that if $\mathbf{X} = (X_0, \dots, X_n)$, then $\tilde{\mathbf{X}} = (X_1, \dots, X_n)$. In order to prove that ν is an isomorphism of algebras, it is enough to prove that

$$\nu[(t^m \otimes X_0^i A(\tilde{\mathbf{X}}))(t^\ell \otimes X_0^j B(\tilde{\mathbf{X}}))] = \nu[t^m \otimes X_0^i A(\tilde{\mathbf{X}})] \nu[t^\ell \otimes X_0^j B(\tilde{\mathbf{X}})] \quad (4-2-5)$$

for any $m, \ell \in \mathbb{Z}, i, j \in \mathbb{N}$ and for any $A, B \in \mathbb{C}[X_1, \dots, X_n]$, where the first product is defined by (4-2-3) in $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[X_0, \dots, X_n]$ and the second one is the usual product in the tensor algebra $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_n]$.

We will need the following lemma whose easy proof by induction is left to the reader.

Lemma 4.2.2. *For any $i, j \in \mathbb{N}, \ell \in \mathbb{Z}$:*

$$(t \frac{d}{dt})^i t^\ell (t \frac{d}{dt})^j = \sum_{p=0}^i \binom{i}{p} \ell^{i-p} t^\ell (t \frac{d}{dt})^{p+j}.$$

We have then:

$$\begin{aligned} \nu[(t^m \otimes X_0^i A(\tilde{\mathbf{X}}))(t^\ell \otimes X_0^j B(\tilde{\mathbf{X}}))] &= \nu[t^{m+\ell} \otimes (X_0 + \ell)^i X_0^j A(\tilde{\mathbf{X}}) B(\tilde{\mathbf{X}})] \\ &= \sum_{p=0}^i \nu(t^{m+\ell} \otimes \binom{i}{p} \ell^{i-p} X_0^{p+j} A(\tilde{\mathbf{X}}) B(\tilde{\mathbf{X}})) \\ &= \sum_{p=0}^i t^{m+\ell} \binom{i}{p} \ell^{i-p} (t \frac{d}{dt})^{p+j} \otimes A(\tilde{\mathbf{X}}) B(\tilde{\mathbf{X}}) \end{aligned}$$

On the other hand we have:

$\nu[t^m \otimes X_0^i A(\tilde{\mathbf{X}})] \nu[t^\ell \otimes X_0^j B(\tilde{\mathbf{X}})] = t^m (t \frac{d}{dt})^i t^\ell (t \frac{d}{dt})^j \otimes A(\tilde{\mathbf{X}}) B(\tilde{\mathbf{X}})$ and then using Lemma 4.2.2 we get:

$$\sum_{p=0}^i t^{m+\ell} \binom{i}{p} \ell^{i-p} (t \frac{d}{dt})^{p+j} \otimes A(\tilde{\mathbf{X}}) B(\tilde{\mathbf{X}}).$$

□

The algebra $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_n]$ can be viewed as the algebra of differential operators on the torus \mathbb{C}^* with coefficients in $\mathbb{C}[X_1, \dots, X_n]$. In other words any element of this algebra can be written as a finite sum of the form:

$$\sum A_{r,s}(\tilde{\mathbf{X}}) t^r (t \frac{d}{dt})^s \quad \text{or} \quad \sum B_{r,s}(\tilde{\mathbf{X}}) t^r (\frac{d}{dt})^s \quad (4-2-6)$$

where $r \in \mathbb{Z}$, $s \in \mathbb{N}$ and $A_{r,s}, B_{r,s} \in \mathbb{C}[X_1, \dots, X_n]$.

Relations (4-1-1), (4-1-2) and (4-1-3) imply then easily the following Corollary.

Corollary 4.2.3. *The Lie algebra \mathfrak{a} is isomorphic to the Lie subalgebra of $\mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_n]$ generated by*

$$b_E(t \frac{d}{dt}, X_1, \dots, X_n) = (n+1)t \frac{d}{dt} + nX_1 + \dots + X_n,$$

$$t \text{ and } t^{-1} b_Y(t \frac{d}{dt}, X_1, \dots, X_n) = t^{-1} \prod_{j=0}^n (t \frac{d}{dt} + X_1 + \dots + X_j + j \frac{d}{2}).$$

Similarly \mathcal{A} is isomorphic to the associative subalgebra generated by these generators and \mathcal{T} is isomorphic to the associative subalgebra generated by $b_E(t \frac{d}{dt}, X_1, \dots, X_n)$, t , $t^{-1} b_Y(t \frac{d}{dt}, X_1, \dots, X_n)$ and t^{-1} .

Definition 4.2.4. *An element $D \in \mathbf{D}(\Omega^+)$ is said to have a radial component if there exists an operator $r_D \in \mathbf{D}(\mathbb{C}^*)$ such that for any $f \in \mathbb{C}[\mathbb{C}^*]$ one has $D(f \circ \Delta_0) = r_D(f) \circ \Delta_0$. The operator r_D is then called the radial component of D .*

Proposition 4.2.5. *The operators E , X , X^{-1} and Y have radial components which are given respectively by*

$$r_E = t \frac{d}{dt}, r_X = t, r_{X^{-1}} = \frac{1}{t}$$

$$r_Y = \frac{1}{t} b_Y(t \frac{d}{dt}, 0, \dots, 0) = \frac{1}{t} \prod_{j=0}^n (t \frac{d}{dt} + j \frac{d}{2})$$

Proof. Only the formula for r_Y needs a proof. The Proposition 4.2.1 gives an algebra embedding

$$\Psi = \varphi \circ \nu^{-1} : \mathbb{C}[t, t^{-1}, t \frac{d}{dt}] \otimes \mathbb{C}[X_1, \dots, X_n] \longrightarrow \text{End}(\mathbb{C}[\Omega^+]).$$

It is easy to see that if $A \in \mathbb{C}[X_1, \dots, X_n]$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$, then the operator $\Psi(A(\tilde{\mathbf{X}})t^m(t\frac{d}{dt})^n)$ can be described as follows:

for $a_0 \in \mathbb{Z}$ and $Q_{\tilde{\mathbf{a}}} \in V_{\tilde{\mathbf{a}}}$, we have

$$\Psi(A(\tilde{\mathbf{X}})t^m(t\frac{d}{dt})^n)\Delta_0^{a_0}Q_{\tilde{\mathbf{a}}} = A(\tilde{\mathbf{a}})a_0^n\Delta_0^{a_0+m}Q_{\tilde{\mathbf{a}}} = A(\tilde{\mathbf{a}})[t^m(t\frac{d}{dt})^n(t^{a_0}) \circ \Delta_0]Q_{\tilde{\mathbf{a}}} \quad (4-2-7)$$

From Corollary 4.2.3 one gets that $Y = \Psi(t^{-1}b_Y(t\frac{d}{dt}, X_1, \dots, X_n))$. Therefore, as $\Delta_0 \in V_{(m,0,\dots,0)}$, we deduce from (4-2-7) that

$$Y\Delta_0^m = [\frac{1}{t}b_Y(t\frac{d}{dt}, 0, \dots, 0)(t^m)] \circ \Delta_0.$$

This gives $r_Y = \frac{1}{t} \prod_{j=0}^n (t\frac{d}{dt} + j\frac{d}{2})$. □

Remark 4.2.6. In the A_{2k-1} case (notation as in Table 1) the relative invariant Δ_0 is the determinant of a $k \times k$ matrix, and $Y = \det(\partial)$. The radial component has been calculated by Raïs ([Ra], page 22). He obtained that $r_Y = \prod_{j=2}^k (t\frac{d}{dt} + j)\frac{d}{dt}$, whereas the preceding formula leads to $r_Y = \frac{1}{t} \prod_{j=0}^{k-1} (t\frac{d}{dt} + j)$ (in that case $\frac{d}{2} = 1$). A simple computation shows that these two differential operators are the same.

5. SEVERAL ALGEBRAS OF INVARIANT DIFFERENTIAL OPERATORS.

5.1. A first result.

We are in a situation where two different groups, G and G' , act on two affine varieties, namely Ω^+ and V^+ . This gives rise to the following four algebras of invariant differential operators:

$$\mathbf{D}(\Omega^+)^G, \quad \mathbf{D}(\Omega^+)^{G'}, \quad \mathbf{D}(V^+)^G, \quad \mathbf{D}(V^+)^{G'}.$$

Proposition 5.1.1. *Every G -invariant differential operator on Ω^+ has polynomial coefficients, i.e. $\mathbf{D}(\Omega^+)^G = \mathbf{D}(V^+)^G$.*

Proof. This result is well known, even in the C^∞ context (see for example [Y], Remark 2, and [No]). It is usually proved by exhibiting a set of generators of $\mathbf{D}(\Omega^+)^G$ which have polynomial coefficients. We give here a direct proof. Recall from (3-1-2) and (3-1-3) the decompositions of $\mathbb{C}[V^+]$ and $\mathbb{C}[\Omega^+]$ into irreducible representations of G :

$$\mathbb{C}[V^+] = \bigoplus_{\mathbf{a} \in \mathbb{N}^{n+1}} V_{\mathbf{a}} \quad , \quad \mathbb{C}[\Omega^+] = \bigoplus_{\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^n} V_{\mathbf{a}}$$

If $D \in \mathbf{D}(\Omega^+)^G$, then by Schur's Lemma D maps each $V_{\mathbf{a}}$ into itself. Therefore such a D stabilizes $\mathbb{C}[V^+]$. This implies that $D \in \mathbf{D}(V^+)^G$. □

Among the preceding spaces the following inclusions are obvious:

$$\begin{array}{ccc} \mathbf{D}(V^+)^G & = & \mathbf{D}(\Omega^+)^G \\ \downarrow & & \downarrow \\ \mathbf{D}(V^+)^{G'} & \longrightarrow & \mathbf{D}(\Omega^+)^{G'} \end{array}$$

In this chapter we will give several descriptions of these algebras.

5.2. The Harish-Chandra isomorphism for G/H and a first description of $\mathbf{D}(\Omega^+)^G \simeq \mathbf{D}(G/H)^G$.

As $\Omega^+ \simeq G/H$ is a complex symmetric space it is well known that $\mathbf{D}(\Omega^+)^G \simeq \mathbf{D}(G/H)^G$ is isomorphic to a polynomial algebra through the so-called Harish-Chandra isomorphism.

For the convenience of the reader let us first recall some details of the Harish-Chandra isomorphism for G/H . In fact what is needed here is an algebraic version of this isomorphism because our algebras of differential operators are defined algebraically. It can be easily deduced from the "real analytic case" given in [H-S], Theorem 4.3. part II.

Let \mathfrak{q} be the orthogonal complement of \mathfrak{h} in \mathfrak{g} with respect to the Killing form of \mathfrak{g} . Therefore one has $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. It is known ([B-R], Chap. 5) that $\mathfrak{t} = \sum_{i=0}^n \mathbb{C}H_{\alpha_i}$ is a maximal abelian subspace of \mathfrak{q} .

Let $\tilde{\Sigma} = \Sigma(\tilde{\mathfrak{g}}, \mathfrak{t})$ be the set of roots of $(\tilde{\mathfrak{g}}, \mathfrak{t})$ and let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{t})$ be the set of roots of $(\mathfrak{g}, \mathfrak{t})$. Of course the roots in $\tilde{\Sigma}$ are just the restrictions to \mathfrak{t} of the roots of \tilde{R} . It is possible to define an order on Σ such that if $\alpha \in R^+$, then the restriction $\bar{\alpha} = \alpha|_{\mathfrak{t}}$ belongs to Σ^+ (see [B-R], Chapter 5, for example). The Weyl group of Σ is denoted by W . Let

$$\mathfrak{n}^+ = \sum_{\gamma \in \Sigma^+} \mathfrak{g}^\gamma, \quad \mathfrak{n}^- = \sum_{\gamma \in \Sigma^-} \mathfrak{g}^\gamma \quad (5-2-1)$$

where, as usual, \mathfrak{g}^γ denotes the root space corresponding to γ . Then we have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t} \oplus \mathfrak{n}^- \quad (5-2-2)$$

and therefore, using the Poincaré-Birkhoff-Witt Theorem, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} decomposes as follows:

$$\mathcal{U}(\mathfrak{g}) = S(\mathfrak{t}) \oplus (\mathcal{U}(\mathfrak{g})\mathfrak{h} + \mathfrak{n}^-\mathcal{U}(\mathfrak{g})) \quad (5-2-3)$$

where $S(\mathfrak{t})$ is the symmetric algebra of \mathfrak{t} . Let us denote by γ' the projection from $\mathcal{U}(\mathfrak{g})$ onto $S(\mathfrak{t})$ defined by the decomposition (5-2-3).

Let $\mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ be the space of elements in $\mathcal{U}(\mathfrak{g})$ which commute with \mathfrak{h} . It is known that γ' is a surjective algebra homomorphism from $\mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ onto $S(\mathfrak{t})$ ([D] 7.4.3.).

The space $S(\mathfrak{t})$ is canonically identified with the space of polynomial functions on \mathfrak{t}^* . Let $\rho = \frac{1}{2} \sum_{\lambda \in \Sigma^-} \lambda$ and define for any $\Lambda \in \mathfrak{t}^*$ and any $z \in \mathcal{U}(\mathfrak{g})$:

$$\gamma(z)(\Lambda) = \gamma'(z)(\Lambda - \rho) \quad (5-2-4)$$

The map γ factorizes through $\mathcal{U}(\mathfrak{g})\mathfrak{h} \cap \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ and defines an isomorphism of algebras (still denoted by γ):

$$\gamma : \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} / \mathcal{U}(\mathfrak{g})\mathfrak{h} \cap \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \longrightarrow S(\mathfrak{t})^W \quad (5-2-5)$$

where $S(\mathfrak{t})^W$ is the algebra of W invariants in $S(\mathfrak{t})$.

This isomorphism is called the *Harish-Chandra isomorphism*.

On the other hand, if $X \in \mathfrak{g}$ and $\varphi \in \mathbb{C}[G]$, one defines an element $\tilde{X} \in \mathbf{D}(G)^{\ell G}$ (the left invariant differential operators on G) by:

$$\forall g \in G \quad \tilde{X}\varphi(g) = \frac{d}{dt}\varphi(g \exp tX)|_{t=0} \quad (5-2-6)$$

The map $X \mapsto \tilde{X}$ extends to a map $U \mapsto \tilde{U}$ from $\mathcal{U}(\mathfrak{g})$ to $\mathbf{D}(G)^{\ell G}$.

For $f \in \mathbb{C}[G/H] = \mathbb{C}[\Omega^+]$ put $\tilde{f} = f \circ \pi$ where $\pi : G \rightarrow G/H$ is the canonical projection. The map $f \mapsto \tilde{f}$ is then an isomorphism from $\mathbb{C}[G/H]$ onto $\mathbb{C}[G]^H$. Let us denote by $\varphi \mapsto \tilde{\varphi}$ the inverse mapping from $\mathbb{C}[G]^H$ onto $\mathbb{C}[G/H]$ given for $g \in G$ by $\tilde{\varphi}(g) = \varphi(g)$.

It is easy to verify that if $U \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ and if $f \in \mathbb{C}[G/H]$, then $\tilde{U}\tilde{f} \in \mathbb{C}[G]^H$. For $U \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ and $f \in \mathbb{C}[G/H]$, define $D_U \in \mathbf{D}(G/H)$ by:

$$(D_U f)(g) = \tilde{\tilde{U}\tilde{f}}(g) = \tilde{U}\tilde{f}(g)$$

Then it is easy to see that $D_U \in \mathbf{D}(G/H)^G$. Let us call r this map $U \mapsto D_U$. It is well known that the map r again factorizes through $\mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \cap \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ and defines an isomorphism of algebras (still denoted by r):

$$r : \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} / \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \cap \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \longrightarrow \mathbf{D}(G/H)^G \quad (5-2-7)$$

From (5-2-5) and (5-2-7) we deduce the following theorem (which is sometimes also called the *Harish-Chandra Isomorphism* ([H-S], Th. 4.3)):

Theorem 5.2.1. *The map $\gamma : \mathbf{D}(G/H)^G \longrightarrow S(\mathfrak{t})^W$ defined for $\dot{U} \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} / \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \cap \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ by $\gamma(r(\dot{U})) = \gamma(\dot{U})$ (where $\gamma(\dot{U})$ has been defined in (5-2-5)) is an isomorphism of algebras.*

The next proposition gives a way to compute the image of a given element $D \in \mathbf{D}(G/H)^G$ under the Harish-Chandra isomorphism. Put $\mathfrak{p}^- = \mathfrak{t} \oplus \mathfrak{n}^-$. Let Λ be a character of the group $R_{\mathfrak{p}^-} = \exp \mathfrak{t} \cdot \exp \mathfrak{n}^- \subseteq G$. We will also denote by Λ the corresponding infinitesimal character on \mathfrak{t} . Let $f_{\Lambda} \in \mathbb{C}[G/H]$ be a dominant vector with weight Λ . This means that:

$$\forall b \in R_{\mathfrak{p}^-}, \forall \dot{g} \in G/H \quad f_{\Lambda}(b\dot{g}) = \Lambda(b)f_{\Lambda}(\dot{g}) \quad (5-2-8)$$

For $D = r(U) \in \mathbf{D}(G/H)^G$ define $\gamma'(D) = \gamma'(U)$.

Proposition 5.2.2. *We have:*

$$\forall D \in \mathbf{D}(G/H)^G, \forall \dot{g} \in G/H \quad Df_{\Lambda}(\dot{g}) = \gamma'(D)(\Lambda)f_{\Lambda}(\dot{g}) = \gamma(D)(\Lambda + \rho)f_{\Lambda}(\dot{g}).$$

Proof. Although the preceding result is already known in different forms, we give a short proof for the convenience of the reader. As $R_{\mathfrak{p}^-}H$ is open in G , it is enough to prove that

$$\forall b \in R_{\mathfrak{p}^-}, \forall U \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}, \quad D_U f_{\Lambda}(\dot{b}) = \gamma'(U)(\Lambda)f_{\Lambda}(\dot{b}).$$

As $b = na$ with $a \in T = \exp \mathfrak{t}$ and $n \in N^- = \exp \mathfrak{n}^-$ and as both D_U and f_{Λ} are left invariant under N^- , it is enough to prove that

$$\forall a \in A, \quad D_U f_{\Lambda}(\dot{a}) = \gamma'(U)(\Lambda)f_{\Lambda}(\dot{a}).$$

Let $U \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$. Then from (5-2-3) we can write $U = \gamma'(U) + D_1X + YD_2$ where $\gamma'(U) \in S(\mathfrak{t})$, $X \in \mathfrak{h}$, $Y \in \mathfrak{n}^-$ and $D_1, D_2 \in \mathcal{U}(\mathfrak{g})$. Using the invariance properties of \tilde{f}_{Λ} it is easy to see that $\tilde{X}\tilde{f}_{\Lambda} = 0$ and $\tilde{Y}\tilde{D}_2\tilde{f}_{\Lambda}(\dot{a}) = 0$. Therefore

we have $D_U f_\Lambda(\dot{a}) = \widetilde{U} \widetilde{f_\Lambda}(a) = \widetilde{\gamma'(U) f_\Lambda}(a)$, for all $a \in T$. But for $X_0 \in \mathfrak{t}$, it is almost obvious that $\widetilde{X_0} \widetilde{f_\Lambda}(a) = X_0(\Lambda) f_\Lambda(a)$. The proposition follows. \square

For $\ell = 0, \dots, n$ define the differential operators D_ℓ by

$$D_\ell = \Delta_0^{1-\ell} \Delta_0(\partial) \Delta_0^\ell = X^{1-\ell} Y X^\ell \quad (5-2-9)$$

These differential operators were probably first considered by A. Selberg in the case of cones of classical types (see [Ter]). They were also used by Yan [Y] who proves the analogue of the following Theorem for symmetric cones. These results on symmetric cones can also be found in the book by J. Faraut and A. Koranyi ([F-K]).

Theorem 5.2.3. *The operators D_0, \dots, D_n are algebraically independent generators of $\mathbf{D}(\Omega^+)^G$. In other words:*

$$\mathbf{D}(\Omega^+)^G = \mathbb{C}[D_0, \dots, D_n].$$

Proof. First of all it is easy to see that the operators D_ℓ are G -invariant (and therefore they have polynomial coefficients by Proposition 5.1.1. Using now the Harish-Chandra isomorphism γ from Theorem 5.2.1 it is enough to prove that the elements $\gamma(D_0), \gamma(D_1), \dots, \gamma(D_n)$ are linearly independent generators of $S(\mathfrak{t})^W$. We first need the following lemma.

Lemma 5.2.4. *The root system $\Sigma(\widetilde{\mathfrak{g}}, \mathfrak{t})$ is always of type C_{n+1} and the root system $\Sigma(\mathfrak{g}, \mathfrak{t}) = \Sigma$ is always of type A_n .*

Proof. Define

$$\widetilde{E}_{ij}(k, \ell) = \{X \in \widetilde{\mathfrak{g}} \mid [H_{\alpha_i}, X] = kX, [H_{\alpha_j}, X] = \ell X, [H_{\alpha_p}, X] = 0 \text{ if } p \neq i, j\}$$

and

$$\widetilde{E}_i(k) = \{X \in \widetilde{\mathfrak{g}} \mid [H_{\alpha_i}, X] = kX, [H_{\alpha_p}, X] = 0 \text{ if } p \neq i\}.$$

We know from [M-R-S, Lemme 4.1.] that:

$$V^+ = \bigoplus_{i < j} \widetilde{E}_{ij} \oplus \bigoplus_{i=0}^n \widetilde{E}_i(2) \quad (5-2-10)$$

and

$$\mathfrak{g} = \bigoplus_{i < j} \widetilde{E}_{ij}(-1, 1) \oplus \bigoplus_{i=0}^n \widetilde{E}_i(0) \oplus \bigoplus_{i < j} \widetilde{E}_{ij}(1, -1) \quad (5-2-11)$$

Moreover one has $\bigoplus_{i=0}^n \widetilde{E}_i(0) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$. The preceding decompositions show that the spaces $\widetilde{E}_{ij}(1, 1)$ and $\widetilde{E}_i(2)$ are the root spaces of the pair $(\widetilde{\mathfrak{g}}, \mathfrak{t})$. Let $\varepsilon_i = \frac{1}{2}\alpha_i$ be the dual basis of the basis H_{α_i} . Now it is clear that the positive roots of $\Sigma(\widetilde{\mathfrak{g}}, \mathfrak{t})$ are the linear forms $\varepsilon_i + \varepsilon_j$ ($i < j$), $\varepsilon_i - \varepsilon_j$ ($i < j$) and $2\varepsilon_i$. This characterizes the root systems C_{n+1} and A_n . \square

End of the proof of Theorem 5.2.3:

As a corollary the Weyl group W of Σ is isomorphic to the symmetric group S_{n+1} of $n + 1$ variables acting by permutations on the coordinates with respect to the α_i 's. In order to compute $\gamma(D_\ell)$ we need to know the highest weight of the \mathfrak{g} -module $V_{\mathbf{a}}$ with respect to \mathfrak{p}^- . This highest weight has been computed in [R-S-2] (Lemme 3.8 p. 155). The result is as follows (recall that $\mathbf{a} = (a_0, \dots, a_n)$):

$$\Lambda(\mathbf{a}) = -a_0\overline{\alpha_0} - (a_0 + a_1)\overline{\alpha_1} - \dots - (a_0 + \dots + a_n)\overline{\alpha_n} \quad (5-2-12)$$

(where $\overline{\alpha_i}$ denotes the restriction of α_i to \mathfrak{t}).

It is now convenient to make the following change of variables:

$$r_i = \sum_{\ell=0}^i a_\ell \quad (i = 0, \dots, n) \quad \text{and} \quad \mathbf{r} = (r_0, r_1, \dots, r_n).$$

Note that

$$\mathbf{a} \in \mathbb{Z} \times \mathbb{N}^n \Leftrightarrow \mathbf{r} = (r_0, r_1, \dots, r_n) \in \mathbb{Z}^n \quad \text{and} \quad r_0 \leq r_1 \leq r_2 \leq \dots \leq r_n.$$

Let us write $\Lambda(\mathbf{r})$ instead of $\Lambda(\mathbf{a})$. Then

$$\Lambda(\mathbf{r}) = - \sum_{i=0}^n r_i \overline{\alpha_i} \quad (5-2-13)$$

We need also to compute ρ . This computation again has already been made in [R-S-2], (Lemme 3.9. p. 155). The result is the following:

$$\rho = \frac{d}{4} \sum_{i < j} (\overline{\alpha_i} - \overline{\alpha_j}) = \frac{d}{4} \sum_{i=0}^n (n - 2i) \overline{\alpha_i} \quad (5-2-14)$$

On the other hand a simple computation shows that:

$$b_{D_\ell}(s_0, \dots, s_n) = b_Y(s_0 + \ell, s_1, \dots, s_n) = \prod_{i=0}^n (s_0 + \ell + s_1 + \dots + s_i + i \frac{d}{2}) \quad (5-2-15)$$

From (5-2-15), (5-2-14) and Prop. 5.2.2 we get

$$\gamma'(D_\ell)(\Lambda(\mathbf{r})) = b_{D_\ell}(\mathbf{a}) = \prod_{i=0}^n (r_i + \ell + i \frac{d}{2}) \quad (5-2-16)$$

Hence, from (5-2-13), (5-2-14) and (5-2-16):

$$\begin{aligned} \gamma(D_\ell)(\Lambda(\mathbf{r})) &= \gamma'(D_\ell)(\Lambda(\mathbf{r}) - \rho) = \gamma'(D_\ell)(- \sum_{i=0}^n r_i \overline{\alpha_i} - \frac{d}{4} \sum_{i=0}^n (n - 2i) \overline{\alpha_i}) \\ &= \gamma'(D_\ell)(\sum_{i=0}^n (-r_i - \frac{d}{4}n + \frac{d}{2}i) \overline{\alpha_i}) \\ &= \prod_{i=0}^n (r_i + \frac{d}{4}n + \ell). \end{aligned} \quad (5-2-17)$$

As expected by Theorem 5.2.3 the polynomials $\gamma(D_\ell)(r_0, \dots, r_n) = \prod_{i=0}^n (r_i + \frac{d}{4}n + \ell)$ are invariant under S_{n+1} . Moreover it is easy to prove (and well

known) that these polynomials are algebraically independent generators of the algebra of symmetric polynomials. Thus Theorem 4.2.3. is proved. \square

Let us note the following corollary of the proof.

Corollary 5.2.5. *For any $D \in \mathcal{T}_0$, let $b_D(\mathbf{r})$ be the polynomial in the \mathbf{r} variable defined by b_D . Let $\rho = \frac{d}{4}(-n, -n+2, \dots, n)$. Then*

$$\gamma(D)(\mathbf{r}) = b_D(\mathbf{r} - \rho).$$

Remark 5.2.6. As $E \in \mathbf{D}(\Omega^+)^G$, we deduce from Theorem 5.2.3 that $\mathcal{T} = \mathbb{C}[E, X, X^{-1}, Y] = \mathbb{C}[X, X^{-1}, Y]$ (E is already a polynomial in X, X^{-1}, Y). In fact this is also a consequence of Théorème 1.1. in [R-S-2].

5.3. Connections with \mathcal{T} and \mathcal{T}_0 .

We will obtain in this section various descriptions of \mathcal{T} and \mathcal{T}_0 in terms of invariant differential operators, and also a characterization of $\mathbf{D}(V^+)^{G'}$ (see Theorem 5.3.3 below).

Theorem 5.3.1.

- 1 - $\mathcal{T}_0 = \mathbf{D}(V^+)^G = \mathbf{D}(\Omega^+)^G \simeq \mathbf{D}(G/H)^G$
- 2 - $\mathcal{T} = \mathbf{D}(\Omega^+)^{G'}$.

Proof. The equality $\mathbf{D}(V^+)^G = \mathbf{D}(\Omega^+)^G$ has already been proved in Prop. 4.1.1. It is clear from relations (3 - 1 - 1) that the operators in \mathcal{T}_0 are G -invariant. Therefore $\mathcal{T}_0 \subseteq \mathbf{D}(\Omega^+)^G$. The converse inclusion is a consequence of Theorem 4.2.3. Thus the first assertion is proved.

Let $A \in \mathbf{D}(\Omega^+)^{G'}$. Then $A = \sum_{i \in \mathbb{Z}} A_i$ (finite sum), where $[E, A_i] = iA_i$. This means just that A_i has global degree i . As E is G' -invariant, it is clear that each A_i is also G' -invariant. Let $\mathbf{a} \in \mathbb{N}^{n+1}$ be such that the restriction of A_i to $V_{\mathbf{a}}$ is non zero. From the G' -invariance we deduce that there exists $\ell \in \mathbb{Z}$ such that the operator A_i maps $V_{\mathbf{a}}$ into $\oplus_{\ell \in \mathbb{Z}} V_{\mathbf{a}+\ell}$, and then, for degree reasons, the components A_i are equal to zero unless $i = (n+1)\ell$, $\ell \in \mathbb{Z}$ and $A = \sum_{i \in \mathbb{Z}} A_{i(n+1)}$. Then the operator $\Delta_0^{-i} A_{i(n+1)} = X^{-i} A_{i(n+1)}$ is G' -invariant and verifies $[E, A_{i(n+1)}] = 0$. As the Euler operator is the infinitesimal generator of the center of G , we obtain that $X^{-i} A_{i(n+1)} \in \mathbf{D}(\Omega^+)^G = \mathbb{C}[D_0, \dots, D_n]$ (Theorem 5.2.3). Thus each $A_{i(n+1)}$ is a polynomial in the operators of the form $X^m Y X^p$, and hence $A \in \mathbb{C}[E, X, X^{-1}, Y] = \mathcal{T}$. \square

As $V_{\mathbf{a}}$ is a G -irreducible module, it is well known that the tensor G -module $V_{\mathbf{a}} \otimes V_{\mathbf{a}}^*$ contains up to constant, a unique G -invariant vector $R_{\mathbf{a}}$ (see for example [H-U]). Moreover as $\mathbb{C}[V^+] \otimes \mathbb{C}[V^+]^*$ is G -isomorphic to $\mathbf{D}(V^+)$, the element $R_{\mathbf{a}}$ can be viewed as a G -invariant differential operator with polynomial coefficients. The operators $R_{\mathbf{a}}$ are sometimes called *Capelli Operators*. Moreover the family of elements $R_{\mathbf{a}}$ ($\mathbf{a} \in \mathbb{N}^{n+1}$) is a vector basis of the vector space $\mathbf{D}(V^+)^G = \mathbf{D}(\Omega^+)^G$.

Another description of the algebra $\mathbf{D}(\Omega^+)^G$, which is due to Zhimin Yan is as follows.

Theorem 5.3.2. Let $\mathbf{1}_j = (0, \dots, 0, 1, 0, \dots, 0)$ where 1 is at the j -th place. The elements $XY = R_{\mathbf{1}_0}, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}$ are algebraically independent generators of $\mathbf{D}(\Omega^+)^G$. Therefore

$$\mathcal{T}_0 = \mathbf{D}(\Omega^+)^G = \mathbf{D}(V^+)^G = \mathbb{C}[XY, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}].$$

Proof. This is just a complex version of Theorem 1.9 of [Y]. \square

Let $\mathbb{C}[X, Y, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}]$ be the subalgebra of $\mathbf{D}(\Omega^+)$ generated by the variables $X, Y, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}$. Note that the operators X and Y do not commute, and also that they do not commute with $R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}$.

Theorem 5.3.3. Denote by $\mathcal{T}_0[X, Y]$ the subalgebra of $\mathbf{D}(\Omega^+)$ generated by \mathcal{T}_0 , X and Y . One has:

$$\mathbf{D}(V^+)^{G'} = \mathbb{C}[X, Y, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}] = \mathcal{T}_0[X, Y]$$

Proof. The inclusion $\mathbb{C}[X, Y, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}] \subset \mathbf{D}(V^+)^{G'}$ is obvious.

Remember that the decomposition of $\mathbb{C}[V^+]$ into irreducible G' -modules is as follows:

$$\mathbb{C}[V^+] = \bigoplus_{\tilde{\mathbf{a}} \in \mathbb{N}^n} \left(\bigoplus_{a_0 \geq 0} \Delta_0^{a_0} V_{\tilde{\mathbf{a}}} \right)$$

Here the various $\Delta_0^{a_0} V_{\tilde{\mathbf{a}}}$ with the same $\tilde{\mathbf{a}}$ are G' -irreducible and G' -isomorphic, but if $\tilde{\mathbf{a}} \neq \tilde{\mathbf{b}}$, then the G' -modules $\Delta_0^{a_0} V_{\tilde{\mathbf{a}}}$ and $\Delta_0^{b_0} V_{\tilde{\mathbf{b}}}$ are non isomorphic. In other words the space $\bigoplus_{a_0 \geq 0} \Delta_0^{a_0} V_{\tilde{\mathbf{a}}}$ is the isotypic component of the irreducible (harmonic) G' -module $V_{\tilde{\mathbf{a}}} = \mathcal{H}_{\tilde{\mathbf{a}}}$ (see (3 – 1 – 5)).

The dual module of $\Delta_0^{a_0} V_{\tilde{\mathbf{a}}}$ can naturally be identified with $\Delta_0^{-a_0} V_{\tilde{\mathbf{a}}}^* \subset \mathbb{C}[V^+]^*$. Then we know (see the above definition of the $R_{\tilde{\mathbf{a}}}$'s) that the submodule $\Delta_0^i V_{\tilde{\mathbf{a}}} \otimes \Delta_0^{-j} V_{\tilde{\mathbf{a}}}^* \subset D(V^+)^{G'}$ contains a unique G' -invariant element namely $X^i R_{\tilde{\mathbf{a}}} Y^j$.

Moreover the set of elements $X^i R_{\tilde{\mathbf{a}}} Y^j$ ($i \geq 0, j \geq 0, \tilde{\mathbf{a}} \in \mathbb{N}^n$) is a vector basis of the space $\mathbf{D}(V^+)^{G'}$. From the preceding theorem each $R_{\tilde{\mathbf{a}}}$ is a polynomial in $R_{\mathbf{1}_0} = XY, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}$. Therefore $\mathbf{D}(V^+)^{G'} \subset \mathbb{C}[X, Y, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}]$ and hence

$$\mathbf{D}(V^+)^{G'} = \mathbb{C}[X, Y, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}].$$

Obviously we have also the inclusion $\mathcal{T}_0[X, Y] \subset \mathbf{D}(V^+)^{G'}$. On the other hand, as the operators $R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}$ are G -invariant, they belong to \mathcal{T}_0 . Therefore $\mathbf{D}(V^+)^{G'} = \mathbb{C}[X, Y, R_{\mathbf{1}_1}, R_{\mathbf{1}_2}, \dots, R_{\mathbf{1}_n}] \subset \mathcal{T}_0[X, Y]$. This completes the proof. \square

Remark 5.3.4. Note first that in all cases $R_{\mathbf{1}_n} = E$. In the special case where $G \simeq SO(k) \times \mathbb{C}^*$ and $V^+ \simeq \mathbb{C}^k$, the preceding theorem yields

$$\mathbf{D}(\mathbb{C}^k)^{SO(k)} = \mathbb{C}[Q(x), Q(\partial), E]$$

where $Q(x) = X = \sum_{i=1}^k x_i^2$, $Q(\partial) = Y = \sum_{i=1}^k \frac{\partial^2}{\partial x_i^2}$.

This was proved by S. Rallis and G. Schiffmann ([Ra-S], Lemma 5.2. p. 112).

6. MORE STRUCTURE

 6.1. The automorphism τ .

Definition 6.1.1. *The automorphism τ of \mathcal{T} is defined by:*

$$\forall D \in \mathcal{T}, \quad \tau(D) = XDX^{-1} \quad (6-1-1)$$

Proposition 6.1.2. *The algebra \mathcal{T}_0 is stable under τ and for any $R \in \mathcal{T}_0$ one has:*

$$XR = \tau(R)X \quad (6-1-2)$$

$$RY = Y\tau(R) \quad (6-1-3)$$

Proof. If $D \in \mathcal{T}_0$, then for homogeneity reasons one has $\tau(D) = XDX^{-1} \in \mathcal{T}_0$. As $XR = XRX^{-1}X$, the identity (6-1-2) is obvious.

We will now prove that (6-1-3) holds on each subspace $V_{\mathbf{a}}$. Let b_R the Bernstein-Sato polynomial of R . Then an easy calculation shows that the left and right hand side of (6-1-3) act on $V_{\mathbf{a}}$ as $b_R(\mathbf{a}-1)b_Y(\mathbf{a})X^{-1}$. \square

Proposition 6.1.3.

1) *Let $\mathcal{T}_0[X, X^{-1}]$ be the subalgebra of $\mathbf{D}(\Omega^+)$ generated by \mathcal{T}_0 , X and X^{-1} . One has*

$$\mathcal{T} = \mathcal{T}_0[X, X^{-1}]. \quad (6-1-4)$$

More precisely any element $D \in \mathcal{T}$ can be written uniquely in the form

$$D = \sum_{i \in \mathbb{Z}} u_i X^i \text{ or } D = \sum_{i \in \mathbb{Z}} X^i u_i \quad (\text{finite sums}) \quad (6-1-5)$$

with $u_i \in \mathcal{T}_0$. Therefore \mathcal{T} is a free left and right \mathcal{T}_0 -module.

2) *Any element D in $\mathcal{T}_0[X, Y]$ can be written uniquely in the form*

$$D = \sum_{i > 0} u_i Y^i + \sum_{i \geq 0} v_i X^i \text{ or } D = \sum_{i > 0} Y^i u_i + \sum_{i \geq 0} X^i v_i \quad (\text{finite sums}) \quad (6-1-6)$$

with $u_i, v_i \in \mathcal{T}_0$. Therefore $\mathcal{T}_0[X, Y]$ is a free left and right \mathcal{T}_0 -module.

Proof.

1) Recall from Remark 5.2.6 that $\mathcal{T} = \mathbb{C}[X, X^{-1}, Y]$. Therefore any $D \in \mathcal{T}$ can be written as a linear combination of elements of the form $u = X^{i_1} Y^{k_1} \dots X^{i_\ell} Y^{k_\ell}$ where $i_j \in \mathbb{Z}, k_j \in \mathbb{N}$. If $p = \sum_{j=1}^{\ell} i_j - k_j$, then $u = X^{i_1} Y^{k_1} \dots X^{i_\ell} Y^{k_\ell} X^{-p} X^p = m X^p$ and $m = X^{i_1} Y^{k_1} \dots X^{i_\ell} Y^{k_\ell} X^{-p} \in \mathcal{T}_0$.

Suppose now that $\sum_i u_i X^i = 0$ (finite sum, $u_i \in \mathcal{T}_0$). As $u_i X^i \in \mathcal{T}_i$, this implies that for all i , one has $u_i X^i = 0$. As \mathcal{T} is an integral domain, we have $u_i = 0$.

2) From the definition the elements of $\mathcal{T}_0[X, Y]$ are sum of monomials $M = X^{i_1} R_{k_1} Y^{j_1} X^{i_2} R_{k_2} Y^{j_2} \dots X^{i_k} R_{k_k} Y^{j_k}$ where $i_\ell, j_\ell \geq 0$ and where $R_{k_\ell} \in \mathcal{T}_0$. From Proposition 6.1.2, we obtain that $M = X^\alpha R_\beta Y^\gamma$ with $\alpha, \gamma \in \mathbb{N}$ and $R_\beta \in \mathcal{T}_0$. Suppose now that M is homogeneous of degree $p \geq 0$, then $M = X^\alpha R_\beta Y^\gamma = u X^p$ where $u = X^\alpha R_\beta Y^\gamma X^{-p} \in \mathcal{T}_0$. If M is homogeneous of degree $p < 0$, then $\gamma > -p$ and $M = u Y^{-p}$ where $u = X^\alpha R_\beta Y^{\gamma+p} \in \mathcal{T}_0$. This shows the existence of the decomposition (6-1-6). The uniqueness of the decomposition is proved as in 1).

□

The following corollary is then obvious.

Corollary 6.1.4. *The inclusion $\mathcal{T}_0[X, Y] = \mathbf{D}(V^+)^{G'} \subset \mathcal{T}_0[X, X^{-1}] = \mathbf{D}(\Omega^+)^{G'} = \mathcal{T}$ is strict but these two algebras have the same "positive" part: for all $p \geq 0$ one has $\mathcal{T}_p = \mathcal{T}_0[X, X^{-1}]_p = \mathcal{T}_0 X^p = \mathcal{T}_0[X, Y]_p$.*

6.2. The center of \mathcal{T} .

As in section 5.2., let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ where \mathfrak{h} is the Lie algebra of the generic isotropy group H and \mathfrak{q} is the orthogonal of \mathfrak{h} with respect to the Killing form. Let S_{n+1} be the symmetric group in $n+1$ variables r_0, r_1, \dots, r_n and let $\mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}}$ be the algebra of symmetric polynomials.

Recall from section 5.2. that S_{n+1} is the Weyl group of the root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{t})$ where $\mathfrak{t} = \sum_{i=0}^n \mathbb{C}H_{\alpha_i}$ is a maximal abelian subspace of \mathfrak{q} . Recall also that we have made the change of variables $\mathbf{a} \longleftrightarrow \mathbf{r}$ where $\mathbf{r} = (r_0, r_1, \dots, r_n)$ is defined by $r_i = \sum_{\ell=0}^i a_\ell$. Then the highest weight $\Lambda(\mathbf{r})$ of $V_{\mathbf{a}}$ with respect to \mathfrak{p}^- is given by $\Lambda(\mathbf{r}) = -\sum_{i=0}^n r_i \bar{\alpha}_i$ where $\bar{\alpha}_i$ denotes the restriction of α_i to \mathfrak{t} (see (5-2-13)). More generally any element $\mathbf{r} = (r_0, r_1, \dots, r_n) \in \mathbb{C}^{n+1}$ can be identified with the element $\Lambda(\mathbf{r}) = -\sum_{i=0}^n r_i \bar{\alpha}_i \in \mathfrak{t}^*$. We make then, for any $D \in \mathcal{T}$, the convention that $b_D(\Lambda(\mathbf{r})) = b_D(r_0, r_1, \dots, r_n) = b_D(a_0, a_1, \dots, a_n)$.

Let ρ be the half sum of roots in Σ^- . Recall from (5-2-14) that $\rho = \frac{d}{4} \sum_{i=0}^n (n-2i) \bar{\alpha}_i$. Then, as seen in section 5.2., the Harish-Chandra isomorphism $\gamma : \mathbf{D}(\Omega^+)^G = \mathcal{T}_0 \longrightarrow \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}}$ is given by

$$\forall D \in \mathcal{T}_0, \quad \gamma(D)(r_0, r_1, \dots, r_n) = b_D(\Lambda(\mathbf{r}) - \rho) \quad (6-2-1)$$

Denote also by τ the isomorphism of $\mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}}$ defined for $P \in \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}}$ by

$$\tau(P)(r_0, r_1, \dots, r_n) = P(r_0 + 1, r_1 + 1, \dots, r_n + 1) \quad (6-2-2)$$

Let us denote by $\mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau}$ the algebra of τ -invariant symmetric polynomials.

Recall that we have previously also denoted by τ the conjugation by X in \mathcal{T} (see (6-1-1)). An easy calculation shows that for $D \in \mathcal{T}_0$ we have $b_{\tau^{-1}(D)}(a_0, a_1, \dots, a_n) = b_{X^{-1}DX}(a_0, a_1, \dots, a_n) = b_D(a_0 + 1, a_1, \dots, a_n)$. In the \mathbf{r} variable this gives $b_{\tau^{-1}(D)}(r_0, r_1, \dots, r_n) = b_D(r_0 + 1, r_1 + 1, \dots, r_n + 1) = \tau(b_D)(r_0, r_1, \dots, r_n)$. Therefore the two definitions of τ are coherent in the sense that they correspond under the Harish-Chandra isomorphism. More precisely we have:

$$\forall D \in \mathcal{T}_0, \quad \gamma(\tau^{-1}(D))(\mathbf{r}) = \tau(\gamma(D))(\mathbf{r}) \quad (6-2-3)$$

Lemma 6.2.1. *For an operator $D \in \mathcal{T}_0$ the following conditions are equivalent:*

i) $\tau(D) = D$ (i.e. D commutes with X).

- ii) $\forall \mathbf{a} \in \mathbb{N}^{n+1}$, $b_D(\mathbf{a} + 1) = b_D(\mathbf{a})$ where as in section 3.1., $\mathbf{a} + 1 = (a_0 + 1, a_1, \dots, a_n)$.
- iii) $\gamma(D) \in \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau}$ (i.e. $\gamma(D)$ is τ -invariant).

Proof. This is just a consequence of (6-2-3) and of the discussion before. \square

Theorem 6.2.2. *Let $\mathcal{Z}(\mathcal{T})$ be the center of \mathcal{T} .*

- 1) *Then $D \in \mathcal{Z}(\mathcal{T})$ if and only if $D \in \mathcal{T}_0$ and D commutes with X (i.e. $\tau(D) = D$).*
- 2) *The center of \mathcal{T} is also the center of $\mathcal{T}_0[X, Y]$, i.e. $\mathcal{Z}(\mathcal{T}) = \mathcal{Z}(\mathcal{T}_0[X, Y])$*
- 3) *$\mathcal{Z}(\mathcal{T})$ is also the set of elements D in \mathcal{T}_0 such that $b_D(\mathbf{a})$ does not depend on a_0 .*
- 4) *$\mathcal{Z}(\mathcal{T}) = \gamma^{-1}(\mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau})$.*

Proof. Let $D \in \mathcal{T}$. Using the \mathbb{Z} -gradation we can write $D = \sum D_i$ (finite sum), where $D \in \mathcal{T}_i$. Suppose now that $D \in \mathcal{Z}(\mathcal{T})$. Then $[E, D] = 0 = \sum (n+1)iD_i$. Therefore $D_i = 0$ if $i \neq 0$, thus $D \in \mathcal{T}_0$. Moreover, of course, D commutes with X .

Conversely suppose that $D \in \mathcal{T}_0$ and that $DX = XD$. Then from Prop. 6.1.3 we obtain that D commutes with every element in \mathcal{T} , i.e. $D \in \mathcal{Z}(\mathcal{T})$. The first assertion is proved.

The second assertion is obvious.

The third and the fourth assertions are consequences of the first one and of Lemma 6.2.1 \square

Remark 6.2.3. As a consequence of the preceding theorem it can be noticed that an operator $D \in \mathcal{T}_0$ which commutes with X , automatically commutes with Y .

Lemma 6.2.4. *Let \mathcal{M} be the hyperplane of \mathbb{C}^{n+1} defined by*

$$\mathcal{M} = \{(r_0, r_1, \dots, r_n) \in \mathbb{C}^{n+1} \mid r_0 + r_1 + \dots + r_n = 0\}.$$

Let $I(\mathcal{M}) = \{P \in \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}} \mid P|_{\mathcal{M}} = 0\}$. Then

$$I(\mathcal{M}) = (r_0 + r_1 + \dots + r_n)\mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}} \text{ and} \\ \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}} = \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau} \oplus I(\mathcal{M}) \quad (6-2-4)$$

Proof. Let $P \in I(\mathcal{M})$. As \mathcal{M} is an irreducible hyperplane defined by the irreducible polynomial $r_0 + r_1 + \dots + r_n$, we have $P = (r_0 + r_1 + \dots + r_n)Q$ where $Q \in \mathbb{C}[r_0, r_1, \dots, r_n]$. As P and $r_0 + r_1 + \dots + r_n$ are S_{n+1} -invariant, the polynomial Q is also S_{n+1} -invariant. Hence

$$I(\mathcal{M}) = (r_0 + r_1 + \dots + r_n)\mathbb{C}[r_0 + r_1 + \dots + r_n]^{S_{n+1}}.$$

Define $F = \mathbb{C} \cdot (1, 1, \dots, 1)$. Then $\mathbb{C}^{n+1} = \mathcal{M} \oplus F$. Let $Q \in \mathbb{C}[\mathcal{M}]^{S_{n+1}}$ be an S_{n+1} -invariant polynomial on \mathcal{M} . Then Q can be extended to a polynomial \tilde{Q} on \mathbb{C}^{n+1} by setting:

$$\forall m \in \mathcal{M}, \forall f \in F \quad \tilde{Q}(m + f) = Q(m) \quad (6-2-5).$$

Then $\tilde{Q} \in \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau}$. In fact $P \longrightarrow P|_{\mathcal{M}}$ is a bijective map from $\mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau}$ onto $\mathbb{C}[\mathcal{M}]^{S_{n+1}}$, whose inverse map is $Q \longrightarrow \tilde{Q}$.

Let $P \in \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau} \cap I(\mathcal{M})$. Then $P|_{\mathcal{M}} = 0$, and as P is also τ -invariant we get $P = 0$. Hence $\mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau} \cap I(\mathcal{M}) = \{0\}$. On the other hand for any $P \in \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}}$ we have $P = \widetilde{P|_{\mathcal{M}}} + (P - \widetilde{P|_{\mathcal{M}}})$. The polynomial $\widetilde{P|_{\mathcal{M}}} \in \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau}$ and the polynomial $(P - \widetilde{P|_{\mathcal{M}}})$ vanishes on \mathcal{M} . Therefore $(P - \widetilde{P|_{\mathcal{M}}}) \in I(\mathcal{M})$. This proves $(6 - 2 - 4)$. \square

An easy induction on the degree of P leads to the following corollary.

Corollary 6.2.5. *Let $P \in \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}}$. Then P can be uniquely written in the form*

$$P(r_0, r_1, \dots, r_n) = \sum_{i=0}^p \alpha_i(r_0, r_1, \dots, r_n)(r_0 + \dots + r_n)^p$$

where $\alpha_i \in \mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}, \tau}$.

Proposition 6.2.6.

$$\mathcal{T}_0 = \mathcal{Z}(\mathcal{T}) \oplus E\mathcal{T}_0 \quad (6 - 2 - 6)$$

Proof. As before let γ be the Harish-Chandra isomorphism between \mathcal{T}_0 and $\mathbb{C}[r_0, r_1, \dots, r_n]^{S_{n+1}}$. As $b_E(\mathbf{a}) = (n+1)a_0 + na_1 + \dots + a_n = r_0 + r_1 + \dots + r_n$, one has $\gamma(E)(r_0 + r_1 + \dots + r_n) = b_E(\Lambda(\mathbf{r}) - \rho) = b_E(-\sum_{i=0}^n (r_i + \frac{d}{4}(n-2i)\overline{\alpha_i})) = \sum_{i=0}^n r_i (= b_E(\mathbf{r}))$. Therefore, using Theorem 6.2.2, the decomposition $(6 - 2 - 6)$ is just the image under γ^{-1} of the decomposition $(6 - 2 - 4)$. \square

Corollary 6.2.7.

1) Let $H \in \mathcal{T}_0$. Then H can be uniquely written in the form:

$$H = H_0 + EH_1 + E^2H_2 + \dots + E^kH_k \text{ where } H_k \in \mathcal{Z}(\mathcal{T})$$

2) Let $D \in \mathcal{T}$, then D can be uniquely written in the form:

$$D = \sum_{k \in \mathbb{Z}, \ell \in \mathbb{N}} H_{k,\ell} E^\ell X^k \text{ or } D = \sum_{k \in \mathbb{Z}, \ell \in \mathbb{N}} H_{k,\ell} X^k E^\ell \text{ (finite sums)}$$

where $H_{k,\ell} \in \mathcal{Z}(\mathcal{T})$

3) Let $D \in \mathcal{T}_0[X, Y]$, then D can be uniquely written in the form:

$$D = \sum_{k \in \mathbb{N}^*, \ell \in \mathbb{N}} H_{k,\ell} E^\ell Y^k + \sum_{r \in \mathbb{N}, s \in \mathbb{N}} H'_{r,s} E^s X^r \text{ (finite sum) or}$$

$$D = \sum_{k \in \mathbb{N}^*, \ell \in \mathbb{N}} H_{k,\ell} Y^k E^\ell + \sum_{r \in \mathbb{N}, s \in \mathbb{N}} H'_{r,s} X^r E^s \text{ (finite sum)}$$

where $H_{k,\ell}, H'_{r,s} \in \mathcal{Z}(\mathcal{T})$

Proof. The first assertion is a direct consequence of Proposition 6.2.6. Assertions 2) and 3) are consequences of 1) and Proposition 6.1.3. \square

Remark 6.2.8. It may be noticed that $(\oplus_{i<0}\mathcal{T}_i) \oplus E\mathcal{T}_0 \oplus (\oplus_{i>0}\mathcal{T}_i)$ is not a subalgebra of \mathcal{T} . Indeed it was shown in [R-S-2] that there exists an operator $\omega_X \in \mathcal{T}_{-1}$ such that $[\omega_X, X] = \frac{k}{2} + \frac{2}{n+1}E$ and this operator does not belong to $E\mathcal{T}_0$.

6.3. Ideals of \mathcal{T} .

Let J be a left (resp. right) ideal of \mathcal{T} . Then J is said to be a graded left (resp. right) ideal if $J = \oplus_{i \in \mathbb{Z}} J_i$ where $J_i = J \cap \mathcal{T}_i$.

Theorem 6.3.1.

1) Let J be a graded left ideal of \mathcal{T} , then $J = \oplus_{i \in \mathbb{Z}} X^i J_0$. Conversely if J_0 is any ideal of the (commutative) algebra \mathcal{T}_0 , then $J = \oplus_{i \in \mathbb{Z}} X^i J_0$ is a graded left ideal of \mathcal{T} .

2) Let J be a graded right ideal of \mathcal{T} , then $J = \oplus_{i \in \mathbb{Z}} J_0 X^i$. Conversely if J_0 is any ideal of the (commutative) algebra \mathcal{T}_0 , then $J = \oplus_{i \in \mathbb{Z}} J_0 X^i$ is a graded right ideal of \mathcal{T} .

3) Let J be a two-sided ideal of \mathcal{T} . Then J is graded, J_0 is a τ -invariant ideal of \mathcal{T}_0 and $J = \oplus_{i \in \mathbb{Z}} X^i J_0 = \oplus_{i \in \mathbb{Z}} J_0 X^i$. Conversely if J_0 is a τ -invariant ideal of \mathcal{T}_0 , then $J = \oplus_{i \in \mathbb{Z}} X^i J_0 = \oplus_{i \in \mathbb{Z}} J_0 X^i$ is a two-sided ideal of \mathcal{T} .

Proof. Let $J = \oplus_{i \in \mathbb{Z}} J_i$ be a graded left ideal. Then J_0 is an ideal of \mathcal{T}_0 and $X^i J_0 \subset J \cap \mathcal{T}_i = J_i$ and conversely $X^{-i} J_i \subset J \cap \mathcal{T}_0 = J_0$. Therefore $J = \oplus_{i \in \mathbb{Z}} X^i J_0$.

If J_0 is an ideal of \mathcal{T}_0 , define $J = \oplus_{i \in \mathbb{Z}} X^i J_0$. Let $D_j \in \mathcal{T}_j$, then $D_j X^i J_0 = X^j X^i (X^{-j} X^{-i} D_j X^i) J_0$. As $(X^{-j} X^{-i} D_j X^i) \in \mathcal{T}_0$, we obtain that $D_j X^i J_0 \in X^{i+j} J_0 \subset J$. Hence J is a graded left ideal.

The proof for graded right ideals is the same.

Let now J be a two-sided ideal. An element $D \in J$ can be written uniquely $D = \sum_{i=-\ell}^{\ell} D_i$, where $D_i \in \mathcal{T}_i$. As $E \in \mathcal{T}_0$, we have $[E, D] = ED - DE \in J$. Moreover $[E, D] = \sum_{i=-\ell}^{\ell} [E, D_i] = \sum_{i=\ell}^{\ell} (n+1)i D_i \in J$. By iterating the bracket with E we get:

$$k = 0, 1, \dots, 2\ell \quad (\text{ad } E)^k D = \sum_{i=-\ell}^{\ell} (n+1)^k i^k D_i \in J \quad (6-3-1)$$

The square matrix defined by the linear system (6-3-1) is invertible because its determinant is Van der Monde. Therefore each operator D_i belongs to J . Hence J is graded.

As J is a two-sided ideal, we have $XJ_0X^{-1} \subset J_0$, and this means that J is τ -invariant.

Applying part 1) and part 2) we see that $J = \oplus_{i \in \mathbb{Z}} X^i J_0 = \oplus_{i \in \mathbb{Z}} J_0 X^i$.

Conversely let J_0 be a τ -invariant ideal of \mathcal{T}_0 . Define $J = \oplus_{i \in \mathbb{Z}} X^i J_0$. According to 1), J is a graded left ideal. But as J_0 is τ -stable one has $X^i J_0 X^{-i} = J_0$. Therefore $J = \oplus_{i \in \mathbb{Z}} (X^i J_0 X^{-i}) X^i = \oplus_{i \in \mathbb{Z}} J_0 X^i$. Then, according to 2), J is also a graded right ideal, hence two-sided. \square

The preceding theorem shows that the two-sided ideals of \mathcal{T} are in one to one correspondence with the τ -invariant ideals of $\mathcal{T}_0 \simeq \mathbb{C}[X_0, X_1, \dots, X_n]^{S_{n+1}}$. For completeness we indicate how such ideals are obtained.

Proposition 6.3.2. *Any ideal of $\mathbb{C}[X_0, X_1, \dots, X_n]^{S_{n+1}}$ which is τ -invariant is generated by a finite number of τ -invariant polynomials.*

Proof. Observe that τ can be defined as an automorphism of the algebra $\mathbb{C}[X_0, X_1, \dots, X_n]$ by the same formula as in (6-2-2). Set $Y_0 = X_0, Y_1 = X_1 - X_0, Y_2 = X_2 - X_1, \dots, Y_n = X_n - X_{n-1}$. Then for any polynomial P we have $(\tau P)(Y_0, \dots, Y_n) = P(Y_0 + 1, Y_1, \dots, Y_n)$. Therefore P is τ -invariant if and only if it depends only on the variables Y_1, \dots, Y_n . Let $P = P_d(Y_1, \dots, Y_n)Y_0^d + P_{d-1}(Y_1, \dots, Y_n)Y_0^{d-1} + \dots + P_0(Y_1, \dots, Y_n)$ be the expansion of P according to the powers of Y_0 in the ring $\mathbb{C}[Y_1, \dots, Y_n][Y_0]$. An easy induction on the degree d shows that P_d is a linear combination of $P, \tau P, \tau^2 P, \dots$. Then by induction the same is true for every coefficient $P_i(Y_1, \dots, Y_n)$ ($0 \leq i \leq d$). Note that the P_i 's are τ -invariant. Consider first an ideal J of $\mathbb{C}[X_0, X_1, \dots, X_n]$ which is (globally) τ -invariant and suppose that $P \in J$. Then from above, we deduce that $P_i \in J$ ($0 \leq i \leq d$).

Let now J_0 be a τ -invariant ideal of $\mathbb{C}[X_0, X_1, \dots, X_n]^{S_{n+1}}$ and put $\sigma_0(X_0, \dots, X_n) = X_0 + X_1 + \dots + X_n$. Suppose that $P \in J_0$. Then $P - P_d \sigma_0^d$ is still in J_0 and its degree in X_0 is strictly less than d . By induction on the degree in X_0 , we find that

$$P = \sum P_j Q_j \quad (6-3-2)$$

where every P_j is in J_0 and is τ -invariant and where every Q_j belongs to $\mathbb{C}[X_0, X_1, \dots, X_n]^{S_{n+1}}$. Hence the collection of all polynomials P_j for all $P \in J_0$ is a set of τ -invariant generators for J_0 . As $\mathbb{C}[X_0, X_1, \dots, X_n]^{S_{n+1}}$ is noetherian, there exists a finite set of such generators. \square

6.4. Noetherianity.

Recall that a non commutative ring \mathcal{R} is said to be noetherian if the right and left ideals are finitely generated, or equivalently if the right and left ideals verify the ascending chain condition (see for example [MC-R]).

Recall also that the rings $\mathcal{T}_0[X], \mathcal{T}_0[X^{-1}], \mathcal{T} = \mathcal{T}_0[X, X^{-1}], \mathcal{T}_0[Y], \mathcal{T}_0[X, Y]$ are defined to be the subrings of $\mathbf{D}(\Omega^+)$ generated by \mathcal{T}_0 and by the elements $X, X^{-1}, \{X, X^{-1}\}, Y$ and $\{X, Y\}$ respectively.

Theorem 6.4.1. *The rings $\mathcal{T}_0[X], \mathcal{T}_0[X^{-1}], \mathcal{T} = \mathcal{T}_0[X, X^{-1}], \mathcal{T}_0[Y], \mathcal{T}_0[X, Y]$ are noetherian.*

Proof. Let S be a ring and $\sigma \in \text{Aut} S$. A σ -derivation of S is a linear map $\delta : S \rightarrow S$ such that $\delta(st) = s\delta(t) + \delta(s)\sigma(t)$. Given a σ -derivation, the skew polynomial ring determined by σ and δ is the ring $S[T, \sigma, \delta] := S\langle T \rangle / \{sT - T\sigma(s) - \delta(s) \mid s \in S\}$ (see [MC-R], section 1.2 for details)

Recall from Proposition 6.1.2 that for all $R \in \mathcal{T}_0$ one has $XR = \tau(R)X, YR = \tau^{-1}(R)Y$, where $\tau(R) = XRX^{-1}$. Recall also from Proposition 6.1.3 that any element $D \in \mathcal{T} = \mathcal{T}_0[X, X^{-1}]$ can be written uniquely $D = \sum_{i \in \mathbb{Z}} u_i X^i$, with $u_i \in \mathcal{T}_0$. The same easy argument shows that any element $D \in \mathcal{T}_0[Y]$ can be written uniquely $D = \sum_{i \in \mathbb{Z}} u_i Y^i$ with $u_i \in \mathcal{T}_0$.

These remarks imply that the rings $\mathcal{T}_0[X]$, $\mathcal{T}_0[X^{-1}]$ and $\mathcal{T}_0[Y]$ (which are subrings of $\mathbf{D}(\Omega^+)$) are respectively isomorphic to the "abstract" skew polynomial rings $\mathcal{T}_0[T, \tau, 0]$ and $\mathcal{T}_0[T, \tau^{-1}, 0]$. They are therefore noetherian by Theorem 1.2.9. of [MC-R].

The ring $\mathcal{T} = \mathcal{T}_0[X, X^{-1}]$ is similarly isomorphic to the skew algebra of Laurent polynomials $\mathcal{T}_0[T, T^{-1}, \tau]$ and is therefore noetherian by Theorem 1.4.5. of [MC-R].

The relations $XR = \tau(R)X, YR = \tau^{-1}(R)Y$, where $R \in \mathcal{T}_0$, imply that $\mathcal{T}_0X + \mathcal{T}_0 = X\mathcal{T}_0 + \mathcal{T}_0$. Moreover $[X, Y] \in \mathcal{T}_0$. These remarks imply that $\mathcal{T}_0[X, Y]$ is an almost normalizing extension of \mathcal{T}_0 in the sense of [MC-R] (section 1.6.10.). As \mathcal{T}_0 is noetherian, this implies by Theorem 1.6.14. of [MC-R], that $\mathcal{T}_0[X, Y]$ is noetherian. \square

6.5. Gelfand-Kirillov dimension.

We will denote by $GK.\dim(\mathcal{R})$ the Gelfand-Kirillov dimension of the algebra \mathcal{R} .

Theorem 6.5.1. *One has $GK.\dim(\mathcal{T}) = GK.\dim(\mathcal{T}_0[X]) = GK.\dim(\mathcal{T}_0[X^{-1}]) = GK.\dim(\mathcal{T}_0[Y]) = GK.\dim(\mathcal{T}_0[X, Y]) = n + 2$.*

Proof. We have seen in the proof of Theorem 6.4.1 that the algebras $\mathcal{T}_0[X]$, $\mathcal{T}_0[X^{-1}]$, $\mathcal{T}_0[Y]$ are isomorphic to the skew polynomial algebra $\mathcal{T}_0[T, \tau]$ (or $\mathcal{T}_0[T, \tau^{-1}]$) and that the algebra $\mathcal{T}_0[X, X^{-1}]$ is isomorphic to the skew algebra of Laurent polynomials $\mathcal{T}_0[T, T^{-1}, \tau]$.

An automorphism ν of \mathcal{T}_0 is called locally algebraic if for any $D \in \mathcal{T}_0$, the set $\{\nu^n(D), n \in \mathbb{N}\}$ spans a finite dimensional vector space. We know from [L-M-O] (Prop.1) that if τ is locally algebraic then $GK.\dim(\mathcal{T}_0[T, \tau]) = GK.\dim(\mathcal{T}_0[T, T^{-1}, \tau]) = GK.\dim(\mathcal{T}_0) + 1$ (see also [Z]).

Let us prove that τ is locally algebraic in the preceding sense. The elements $D \in \mathcal{T}_0$ are in one to one correspondence with their Bernstein-Sato polynomial b_D . An easy computation shows that $b_{\tau(D)}(\mathbf{a}) = b_{XD^{-1}}(\mathbf{a}) = b_D(\mathbf{a} - 1)$. Therefore the Bernstein-Sato polynomials $b_{\tau^n(D)}$ have the same degree as b_D . Hence the space spanned by the family $b_{\tau^n(D)}$ is finite dimensional, and τ is locally algebraic.

As \mathcal{T}_0 is a polynomial algebra in $n + 1$ variables (see Theorem 5.2.3) we have $GK.\dim(\mathcal{T}_0) = n + 1$ (see for example [MC-R], Prop. 8.1.15. p. 282). Therefore $GK.\dim(\mathcal{T}_0[X]) = GK.\dim(\mathcal{T}_0[X^{-1}]) = GK.\dim(\mathcal{T}_0[Y]) = GK.\dim(\mathcal{T}_0[X, X^{-1}]) = n + 2$.

As $\mathcal{T}_0[X] \subset \mathcal{T}_0[X, Y] \subset \mathcal{T}_0[X, X^{-1}]$, we have also $GK.\dim(\mathcal{T}_0[X, Y]) = n + 2$. \square

7. GENERATORS AND RELATIONS FOR $\mathcal{T}_0[X, Y]$.

7.1. Generalized Smith algebras.

Let \mathbf{R} be a commutative associative algebra over \mathbb{C} with unit element 1 and without zero divisors. Let $f \in \mathbf{R}[t]$ be a polynomial in one variable with

coefficients in \mathbf{R} and let $n \in \mathbb{N}$. In any associative algebra A the bracket of two elements a and b is defined by $[a, b] = ab - ba$.

Definition 7.1.1. *The (generalized) Smith algebra $S(\mathbf{R}, f, n)$ is the associative algebra over \mathbf{R} with generators x, y, e subject to the relations*

$$[e, x] = (n+1)x, \quad [e, y] = -(n+1)y, \quad [y, x] = f(e).$$

Remark 7.1.2.

1) In the case $\mathbf{R} = \mathbb{C}$ and $n = 0$, the algebras $S(\mathbb{C}, f, 0)$ were introduced by Paul Smith in [Sm] as a class of algebras similar to the enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$. He also developed a very interesting representation theory for these algebras (see [Sm]).

2) One can prove, as in [Sm], that if the degree of f is 1, and if the leading coefficient is invertible in \mathbf{R} , then $S(\mathbf{R}, f, n)$ is isomorphic to the enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(\mathbf{R}))$.

Proposition 7.1.3. *Let \mathfrak{b} the 2-dimensional Lie algebra over \mathbf{R} , with basis $\{\varepsilon, \alpha\}$ and relation $[\varepsilon, \alpha] = (n+1)\alpha$. Let $\mathcal{U}(\mathfrak{b})$ the enveloping algebra of \mathfrak{b} . Define an automorphism σ of $\mathcal{U}(\mathfrak{b})$ by $\sigma(\alpha) = \alpha$ and $\sigma(\varepsilon) = \varepsilon - (n+1)$ and define also a σ -derivation δ of $\mathcal{U}(\mathfrak{b})$ by $\delta(\alpha) = f(\varepsilon)$ and $\delta(\varepsilon) = 0$. Then $S(\mathbf{R}, f, n) \simeq \mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$ (see the proof of Theorem 6.4.1 for the definition of the skew polynomial algebra $\mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$).*

Proof. The proof is the same as the one given by P. Smith ([Sm], Prop. 1.2.). It suffices to remark that the algebra $\mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$ is an algebra over \mathbf{R} with generators ε, α, t subject to the relations

$$[\varepsilon, \alpha] = (n+1)\alpha,$$

$$\alpha t = t\sigma(\alpha) + \delta(\alpha) \text{ which is equivalent to } \alpha t = t\alpha + f(\varepsilon),$$

$$\varepsilon t = t\sigma(\varepsilon) + \delta(\varepsilon) \text{ which is equivalent to } \varepsilon t = t(\varepsilon - (n+1)) = t\varepsilon - (n+1)t.$$

Then the isomorphism $S(\mathbf{R}, f, n) \simeq \mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$ is given by $e \mapsto \varepsilon$, $x \mapsto \alpha$ and $y \mapsto t$

□

The following Corollary is also analogous to Corollary 1.3. in [Sm] and corresponds to a kind of Poincaré-Birkhoff-Witt Theorem for $S(\mathbf{R}, f, n)$.

Corollary 7.1.4.

$S(\mathbf{R}, f, n)$ is a noetherian domain with \mathbf{R} -basis $\{y^i x^j e^k, i, j, k \in \mathbb{N}\}$ (or any similar family of ordered monomials obtained by permutation of the elements (y, x, e)).

Proof. (compare with [Sm], proof of corollary 1.3 p.288). We know from [C] or [MC-R] Th.1.2.9, that as $\mathcal{U}(\mathfrak{b})$ is a noetherian domain, so is $S(\mathbf{R}, f, n) \simeq \mathcal{U}(\mathfrak{b})[t, \sigma, \delta]$. Since

$$\begin{aligned} \mathcal{U}(\mathfrak{b})[t, \sigma, \delta] &= \mathcal{U}(\mathfrak{b}) \oplus \mathcal{U}(\mathfrak{b})t \oplus \mathcal{U}(\mathfrak{b})t^2 \oplus \mathcal{U}(\mathfrak{b})t^3 \oplus \cdots \oplus \mathcal{U}(\mathfrak{b})t^\ell \oplus \cdots \\ &= \mathcal{U}(\mathfrak{b}) \oplus t\mathcal{U}(\mathfrak{b}) \oplus t^2\mathcal{U}(\mathfrak{b}) \oplus t^3\mathcal{U}(\mathfrak{b}) \oplus \cdots \oplus t^\ell\mathcal{U}(\mathfrak{b}) \oplus \cdots \end{aligned}$$

(direct sums of \mathbf{R} -modules) and since the Poincaré-Birkhoff-Witt Theorem is still true for enveloping algebras of Lie algebras which are free over rings (see [Bou-1]), the ordered monomials in (y, x, e) beginning or ending with

y form a basis of the algebra $S(\mathbf{R}, f, n)$. To obtain the basis $\{e^i y^j x^k\}$ or $\{x^k y^j e^i\}$ it suffices to replace the algebra \mathfrak{b} by the algebra \mathfrak{b}_- which is generated by e and y . \square

Remark 7.1.5. The adjoint action of e ($e \mapsto [e, u]$) on $S(\mathbf{R}, f, n)$ is semi-simple and gives a decomposition of $S(\mathbf{R}, f, n)$ into weight spaces:

$$S(\mathbf{R}, f, n) = \bigoplus_{\nu \in \mathbb{Z}} S(\mathbf{R}, f, n)^\nu$$

where $S(\mathbf{R}, f, n)^\nu = \{u \in S(\mathbf{R}, f, n), [e, u] = \nu(n+1)u\}$. As $[e, x^j y^i e^k] = (n+1)(j-i)y^i x^j e^k$, we obtain, using Corollary 7.1.4, that the ordered monomials of the form $x^i y^j e^k$ form an \mathbf{R} -basis for $S(\mathbf{R}, f, n)^0$. Moreover as $yx = xy + f(e)$, it is easy to see that $S(\mathbf{R}, f, n)^0 = \mathbf{R}[xy, e] = \mathbf{R}[yx, e]$, where $\mathbf{R}[xy, e]$ (resp. $\mathbf{R}[yx, e]$) denotes the \mathbf{R} -subalgebra generated by xy (resp. yx) and e .

Lemma 7.1.6. *There exists an element $u \in \mathbf{R}[t]$, which is unique up to addition of an element of \mathbf{R} , such that*

$$f(t) = u(t+n+1) - u(t) \quad (7-1-1)$$

Proof. Consider the \mathbf{R} -linear map $\Delta : \mathbf{R}[t] \rightarrow \mathbf{R}[t]$ defined by $(\Delta P)(t) = P(t+n+1) - P(t)$. Define $P_1[t] = \frac{1}{n+1}t$, then $(\Delta P_1)(t) = 1$ and therefore the elements of \mathbf{R} belong to the image of Δ . Suppose that the space $\mathbf{R}[t]^k$ of polynomials in $\mathbf{R}[t]$ of degree less than k belong to the image of Δ . Let $P_{k+2}(t) = t^{k+2}$, then $(\Delta P_{k+2})(t) = t^{k+1} \bmod \mathbf{R}[t]^{k+1}$ and therefore we have proved by induction that Δ is surjective. Moreover we see that $\ker(\Delta) = \mathbf{R}$. \square

The following result is similar to Proposition 1.5 in [Sm], one has just to be careful when working over an arbitrary ring, rather than \mathbb{C} . It shows that, analogously to the enveloping algebra of \mathfrak{sl}_2 , there is a Casimir-like element which generates the center of $S(\mathbf{R}, f, n)$ over \mathbf{R} .

Proposition 7.1.7. *Let u be as in the preceding Lemma. Define*

$$\Omega_1 = xy - u(e) \text{ and } \Omega_2 = xy + yx - u(e+n+1) - u(e).$$

Then $\Omega_2 = 2\Omega_1$ and the center of $S(\mathbf{R}, f, n)$ is $\mathbf{R}\Omega_1 = \mathbf{R}\Omega_2$.

Proof. From the defining relations of $S(\mathbf{R}, f, n)$, we have $[y, x] = yx - xy = f(e) = u(e+n+1) - u(e)$, hence $yx = xy + u(e+n+1) - u(e)$ and therefore $\Omega_2 = 2\Omega_1$.

Let us now prove that Ω_1 is central. As $\Omega_1 \in \mathbf{R}[xy, e] = S(\mathbf{R}, f, n)^0$ (Remark 7.1.5), we see that Ω_1 commutes with e .

From the defining relations of $S(\mathbf{R}, f, n)$ we have also $[e, x] = ex - xe = (n+1)x$, hence $ex = x(e+n+1)$ and therefore, for any $k \in \mathbb{N}$, $e^k = x(t+n+1)^k$.

This implies of course that for any polynomial $P \in \mathbf{R}[t]$ we have

$$P(e)x = xP(e+n+1) \text{ or } P(e-n-1)x = xP(e). \quad (7-1-2)$$

Similarly one proves that

$$P(e)y = yP(e - n - 1) \text{ or } P(e + n + 1)y = yP(e). \quad (7 - 1 - 3)$$

Let us now show that Ω_1 commutes with x :

$$\begin{aligned} x\Omega_1 &= x(xy - u(e)) = x^2y - xu(e) = x(yx - u(e + n + 1) + u(e)) - xu(e) \\ &= xyx - xu(e + n + 1) = xyx - u(e)x \text{ (using (7 - 1 - 2))} \\ &= \Omega_1x. \end{aligned}$$

A similar calculation using (7 - 1 - 3) shows that Ω_1 commutes also with y . Hence Ω_1 belongs to the center of $S(\mathbf{R}, f, n)$.

Let now z be a central element of $S(\mathbf{R}, f, n)$. Then $z \in S(\mathbf{R}, f, n)^0$. We have $S(\mathbf{R}, f, n)^0 = \mathbf{R}[xy, e] = \mathbf{R}[\Omega_1, e]$, and hence z can be written as follows:

$$z = \sum c_i(e)\Omega_1^i \quad (\text{finite sum})$$

where $c_i(e) \in \mathbf{R}[e]$.

We have:

$$\begin{aligned} 0 &= [z, x] = [\sum c_i(e)\Omega_1^i, x] = \sum [c_i(e), x]\Omega_1^i \\ &= \sum (c_i(e)x - xc_i(e))\Omega_1^i = \sum x(c_i(e + n + 1) - c_i(e))\Omega_1^i \text{ (using (7-1-2))} \\ &= x(\sum (c_i(e + n + 1) - c_i(e))\Omega_1^i) \end{aligned}$$

As the algebra $S(\mathbf{R}, f, n)$ has no zero divisors we get:

$$\sum (c_i(e + n + 1) - c_i(e))\Omega_1^i = 0 \quad (*)$$

Let us now remark that the elements $e^j\Omega_1^i$ ($i, j \in \mathbb{N}$) are free over \mathbf{R} . Suppose that we have

$$\sum_{i,j} \alpha_{i,j} e^j \Omega_1^i = 0 \quad \text{with } \alpha_{i,j} \in \mathbf{R}.$$

As $\Omega_1 = xy - u(e)$, we have

$$\Omega_1^i = x^i y^i \text{ modulo monomials in } e^k x^p y^p \text{ with } p < i.$$

Therefore for all i, j , we have $\alpha_{i,j} = 0$. Then from (*) and from Corollary 7.1.4 above we obtain $c_i(e + n + 1) - c_i(e) = 0$, for all i, j , and hence $c_i(t + n + 1) - c_i(t) = 0$, for all i, j . From the proof of Lemma 7.1.6, we obtain that $c_i \in \mathbf{R}$, for all i . □

7.2. Some quotients of Generalized Smith Algebras.

Let $u \in \mathbf{R}[t]$ be an arbitrary polynomial with coefficients in \mathbf{R} , and let $n \in \mathbb{N}$.

Definition 7.2.1. *The algebra $U(\mathbf{R}, u, n)$ is the associative algebra over \mathbf{R} with generators $\tilde{x}, \tilde{y}, \tilde{e}$ subject to the relations*

$$[\tilde{e}, \tilde{x}] = (n + 1)\tilde{x}, \quad [\tilde{e}, \tilde{y}] = -(n + 1)\tilde{y}, \quad \tilde{x}\tilde{y} = u(\tilde{e}), \quad \tilde{y}\tilde{x} = u(\tilde{e} + n + 1).$$

Remark 7.2.2. Let $f \in \mathbf{R}[t]$ be defined by $f(t) = u(t + n + 1) - u(t)$ (see Lemma 7.1.6). Then, from the definitions we have:

$$U(\mathbf{R}, u, n) = S(\mathbf{R}, f, n)/(xy - u(e)) = S(\mathbf{R}, f, n)/(\Omega_1)$$

where $(xy - u(e))$ is the ideal (automatically two-sided) generated by $xy - u(e) = \Omega_1$. Again, as for $S(\mathbf{R}, f, n)$, the adjoint action of \tilde{e} gives a decomposition of $U(\mathbf{R}, u, n)$ into weight spaces:

$$U(\mathbf{R}, u, n) = \bigoplus_{\nu \in \mathbb{Z}} U(\mathbf{R}, u, n)^\nu \quad (7 - 2 - 1)$$

where $U(\mathbf{R}, u, n)^\nu = \{\tilde{v} \in U(\mathbf{R}, u, n), [\tilde{e}, \tilde{v}] = \nu(n + 1)\tilde{v}\}$.

Proposition 7.2.3. Let $u \in \mathbf{R}[t]$ and $s \in \mathbb{N}$. The \mathbf{R} -linear mappings

$$\begin{aligned} \varphi : \mathbf{R}[t] &\longrightarrow U(\mathbf{R}, u, n) & \psi : \mathbf{R}[t] &\longrightarrow U(\mathbf{R}, u, n) \\ P &\longmapsto \varphi(P) = \tilde{x}^s P(\tilde{e}) & P &\longmapsto \psi(P) = \tilde{y}^s P(\tilde{e}) \end{aligned}$$

are injective (in particular the subalgebra $\mathbf{R}[\tilde{e}] \subset U(\mathbf{R}, u, n)$ generated by \tilde{e} is a polynomial algebra).

Proof.

Define $f(t) = u(t + n + 1) - u(t)$. Every element of $S(\mathbf{R}, f, n)$ can be written in a unique way under the form

$$\sum a_{k,\ell,m} e^k x^\ell y^m \quad (a_{k,\ell,m} \in \mathbf{R})$$

(Corollary 7.1.4). Therefore, from Remark 7.2.2, every element in $U(\mathbf{R}, u, n)$ can be written under the form:

$$\sum a_{k,\ell,m} \tilde{e}^k \tilde{x}^\ell \tilde{y}^m \quad (a_{k,\ell,m} \in \mathbf{R}).$$

Let $P(t) = \sum_{i=0}^p a_i t^i$, ($a_i \in \mathbf{R}$) be a polynomial such that $\tilde{x}^s P(\tilde{e}) = 0$ (i.e. $P \in \ker \varphi$). As $U(\mathbf{R}, u, n) = S(\mathbf{R}, f, n)/(\Omega_1)$ (Remark 7.2.2), we see that

$$x^s \sum_{i=0}^p a_i e^i \in (\Omega_1) = \{r\Omega_1, r \in S(\mathbf{R}, f, n)\}.$$

Therefore there exists $\alpha \in S(\mathbf{R}, f, n)$ such that

$$x^s \sum_{i=0}^p a_i e^i = \alpha \Omega_1 = \alpha(xy - u(e)).$$

If $\alpha = \sum a_{k,\ell,m} e^k x^\ell y^m$, using the fact that $\Omega_1 = xy - u(e)$ is central and relation (7 - 1 - 2) we get:

$$\begin{aligned} x^s \sum_{i=0}^p a_i e^i &= \left(\sum_{k,\ell,m} a_{k,\ell,m} e^k x^\ell y^m \right) (xy - u(e)) = \sum_{k,\ell,m} a_{k,\ell,m} e^k x^\ell (xy - u(e)) y^m \\ &= \sum_{k,\ell,m} a_{k,\ell,m} e^k x^{\ell+1} y^{m+1} - \sum_{k,\ell,m} a_{k,\ell,m} e^k x^\ell u(e) y^m \\ &= \sum_{k,\ell,m} a_{k,\ell,m} e^k x^{\ell+1} y^{m+1} - \sum_{k,\ell,m} a_{k,\ell,m} e^k u(e - \ell(n+1)) x^\ell y^m (**). \end{aligned}$$

Suppose now that $\alpha \neq 0$, then one can define

$$\ell_0 = \max\{\ell \in \mathbb{N}, \exists k, m, a_{k,\ell,m} \neq 0\}.$$

Let k_0, m_0 be such that $\alpha_{k_0, \ell_0, m_0} \neq 0$. From $(**)$ above we get:

$$x^s \sum_{i=0}^p a_i e^i + \sum_{k, \ell, m} a_{k, \ell, m} e^k u(e - \ell(n+1)) x^\ell y^m = \sum_{k, \ell, m} a_{k, \ell, m} e^k x^{\ell+1} y^{m+1}.$$

Using again $(7-1-2)$ we obtain:

$$\sum_{i=0}^p a_i (e - (n+1)s)^i + \sum_{k, \ell, m} a_{k, \ell, m} e^k u(e - \ell(n+1)) x^\ell y^m = \sum_{k, \ell, m} a_{k, \ell, m} e^k x^{\ell+1} y^{m+1}.$$

The left hand side of the preceding equality does not contain the monomial $e^{k_0} x^{\ell_0+1} y^{m_0+1}$, whereas the right hand side does. As the elements $e^k x^\ell y^m$ are a basis over \mathbf{R} (Corollary 7.1.4), we obtain a contradiction. Therefore $\alpha = 0$, and hence $x^s \sum_{i=0}^p a_i e^i = 0$, and again from Corollary 7.1.4, we obtain that $a_i = 0$ for all i . This proves that $\ker \varphi = \{0\}$. The proof for ψ is similar. \square

Corollary 7.2.4. *Every element \tilde{u} in $U(\mathbf{R}, u, n)$ can be written in a unique way under the form*

$$\tilde{u} = \sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} \tilde{y}^\ell \tilde{e}^k + \sum_{m \geq 0, r \geq 0} \beta_{m, r} \tilde{x}^m \tilde{e}^r$$

with $\alpha_{k, \ell}, \beta_{m, r} \in \mathbf{R}$.

Proof. From Corollary 7.1.4 and Remark 7.2.2 we know that any element in $U(\mathbf{R}, u, n)$ can be written (in a non unique way) as a linear combination, with coefficients in \mathbf{R} , of the elements $\tilde{x}^i \tilde{y}^j \tilde{e}^k$.

Suppose that $i \geq j$. Then we have:

$$\tilde{x}^i \tilde{y}^j \tilde{e}^k = \tilde{x}^{i-j} \tilde{x}^j \tilde{y}^j \tilde{e}^k.$$

As $\tilde{y}\tilde{x} = u(\tilde{e} + n + 1)$ and $\tilde{x}\tilde{y} = u(\tilde{e})$, we see that $\tilde{x}^j \tilde{y}^j = Q_j(\tilde{e})$, where Q_j is a polynomial with coefficients in \mathbf{R} . Therefore $\tilde{x}^i \tilde{y}^j \tilde{e}^k = \sum_{\ell} \gamma_{\ell} \tilde{x}^{i-j} \tilde{e}^{\ell}$, with $\gamma_{\ell} \in \mathbf{R}$. Similarly one can prove that if $i < j$, we have $\tilde{x}^i \tilde{y}^j \tilde{e}^k = \sum_{\ell} \delta_{\ell} \tilde{y}^{j-i} \tilde{e}^{\ell}$, with $\delta_{\ell} \in \mathbf{R}$. This shows that any element \tilde{u} in $U(\mathbf{R}, u, n)$ can be written under the expected form.

Suppose now that:

$$\sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} \tilde{y}^\ell \tilde{e}^k + \sum_{m \geq 0, r \geq 0} \beta_{m, r} \tilde{x}^m \tilde{e}^r = 0.$$

Then, as $\tilde{y}^\ell \tilde{e}^k \in U(\mathbf{R}, u, n)^\ell$ and $\tilde{x}^m \tilde{e}^r \in U(\mathbf{R}, u, n)^m$, we deduce from $(7-2-1)$ that

$$\forall \ell > 0, \sum_k \alpha_{k, \ell} \tilde{y}^\ell \tilde{e}^k = 0, \quad \forall m \geq 0, \sum_r \beta_{m, r} \tilde{x}^m \tilde{e}^r = 0.$$

Then from Proposition 7.2.3, we deduce that $\alpha_{k, \ell} = 0$ and $\beta_{m, r} = 0$. \square

7.3. Generators and relations for $\mathcal{T}_0[X, Y] = \mathbf{D}(V^+)^{G'}$.

We know from Corollary 6.2.7 that any element $H \in \mathcal{T}_0$ can be written uniquely in the form $H = u_H(E)$, where $u_H \in \mathcal{Z}(\mathcal{T})[t]$. In particular $XY = u_{XY}(E)$. As the polynomial u_{XY} will play an important role in the Theorem below, let us emphasize the connection between u_{XY} and the Bernstein polynomial b_Y . First remark that $b_Y = b_{XY}$. Moreover define $b_Y^\rho(\mathbf{r}) = b_Y(\mathbf{r} - \rho)$. We know from section 5.2 that b_Y^ρ is S_{n+1} -invariant and is the image under the the Harish-Chandra isomorphism γ of the the G -invariant differential operator XY . In fact as $XY = D_0$ (see (5 - 2 - 9), we have already compute b_Y^ρ (see (5 - 2 - 17)):

$$b_Y^\rho(\mathbf{r}) = \prod_{i=0}^n (r_i + \frac{d}{4}n)$$

From Corollary 6.2.5 this polynomial can be written uniquely in the form

$$b_Y^\rho(\mathbf{r}) = \sum_j \beta_j(\mathbf{r})(r_0 + r_1 + \cdots + r_n)^j$$

where $\beta_j \in \mathcal{Z}(\mathcal{T})$. Then from section 5.2 we obtain:

Proposition 7.3.1. *Keeping the notations above, we have*

$$u_{XY}(t) = \sum_j \gamma^{-1}(\beta_j)t^j$$

Let us now state the main result of this section:

Theorem 7.3.2. *The mapping*

$$\tilde{x} \mapsto X, \quad \tilde{y} \mapsto Y, \quad \tilde{e} \mapsto E$$

extends uniquely to an isomorphism of $\mathcal{Z}(\mathcal{T})$ -algebras between $U(\mathcal{Z}(\mathcal{T}), u_{XY}, n)$ and $\mathcal{T}_0[X, Y]$.

Proof. As

$$[E, X] = (n+1)X, \quad [E, Y] = -(n+1)Y, \quad XY = u_{XY}(E), \quad YX = u_{XY}(E+n+1),$$

and as $\mathcal{T}_0 = \mathcal{Z}(\mathcal{T})[E] \simeq \mathcal{Z}(\mathcal{T})[t]$, we know from the universal property of $U(\mathcal{Z}(\mathcal{T}), u_{XY}, n)$, that the mapping

$$\tilde{x} \mapsto X, \quad \tilde{y} \mapsto Y, \quad \tilde{e} \mapsto E$$

extends uniquely to a surjective morphism of $\mathcal{Z}(\mathcal{T})$ -algebras:

$$\varphi : U(\mathcal{Z}(\mathcal{T}), u_{XY}, n) \longrightarrow \mathcal{T}_0[X, Y].$$

Let $\tilde{u} = \sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} \tilde{y}^\ell \tilde{e}^k + \sum_{m \geq 0, r \geq 0} \beta_{m, r} \tilde{x}^m \tilde{e}^r \in \ker \varphi$ (with $\alpha_{k, \ell}, \beta_{m, r} \in \mathcal{Z}(\mathcal{T})$, see Corollary 7.2.4). We have

$$\begin{aligned} \varphi(\tilde{u}) &= \varphi\left(\sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} \tilde{y}^\ell \tilde{e}^k + \sum_{m \geq 0, r \geq 0} \beta_{m, r} \tilde{x}^m \tilde{e}^r\right) \\ &= \sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} Y^\ell E^k + \sum_{m \geq 0, r \geq 0} \beta_{m, r} X^m E^r \\ &= 0 \end{aligned}$$

Then Corollary 6.2.7 implies that $\alpha_{k, \ell} = \beta_{m, r} = 0$, hence $\ker \varphi = \{0\}$. \square

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